The manifold of Motions and the total mass of a mechanical system

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February 13, 2014

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In memory of Jean-Marie Souriau (1922–2012)

Motions of a Lagrangian mechanical system

Around 1810-1811, while studying the slow variation of the six orbital elements of a planet in the solar system (half major axis, eccentricity, inclination, ...) due to perturbatons of its Keplerian motion by graviatational interactions with the other planets, Joseph Louis Lagrange (1736–1813) introduced an important new idea: he considered the set of all possible Keplerian (unperturbed) motions of the planet; he proved that locally, this set has the structure of a six-dimensional smooth manifold, on which the orbital elements make a system of local coordinates.

Moreover, he proved that there exists on this manifold a natural symplectic structure.

The slow variation of orbital elements of the planet due to its gravitational interaction with other planets can then be described by a smooth curve, parametrized by the time, drawn on the manifold of unperturbed motions.

Motions of a Lagrangian mechanical system (2)

Lagrange proved that this parametrized curve on the manifold of Keplerian motions obeys a time-dependent differential equation. He explicitely determined that differential equation and proved that its form is that of a Hamilton's equation, with a time-dependent Hamiltonian. At that time the great Irish mathematician *William Rowan Hamilton* (1805–1865) was five or six years old.

Lagrange introduced a composition law on the set of local coordinates on the manifold of Keplerian motions, today called in Mechanics the *Lagrange parentheses* (in modern mathematical language Lagrange parentheses are the components of the natural symplectic form). He generalized his results for all Lagrangian mechanical systems.

At the same time, Siméon Denis Poisson (1781–1840) defined another composition law more convenient than the Lagrange parentheses because it can be applied to any pair of smooth functions on the manifold of motions: the *Poisson bracket*, today widely used in classical and quantum Mechanics.

Motions of a Lagrangian mechanical system (3)

While studying the perturbations of Keplerian motions due to gravitational interactions between the planets, Lagrange and Poisson discovered an important property of the *set of all possible Keplerian (unperturbed) motions of a planet*, and more generally, of the *set of all possible solutions of a Lagrangian (or Hamiltonian) mechanical system*: this set has a natural structure of *symplectic manifold*. It is of this structure that I will speak.

During the years 1960–1970, *Jean-Marie Souriau* formalized Lagrange's results in modern mathematical language and gave them a global formulation. He proved that under very general assumptions, the set of all possible solutions of a classical mechanical system, involving material points interacting by very general forces, has a *smooth manifold structure (not always Hausdorff)* and is endowed with a *natural symplectic form*. He called it the *manifold of motions* of the mechanical system.

The manifold of motion of a mechanical system

Souriau observed that the Lagrangian and Hamiltonian formalisms, in their usual formulations, *involve the choice of a particular reference frame*, in which the motion is described. This choice of a particular reference frame *destroys the natural symmetries of the system*. Examples:

If we use the *Lagrangian formalism* (Lagrange's equations of motion or Hamilton's principle of least action), the Lagrangian of the system depends on the reference frame: it is not invariant with respect to Galilean transformations.

If we use the *Hamiltonian formalism* (Hamilton's equations), the Hamiltonian of the system depends on the reference frame: as the Lagrangian, it is not invariant with respect to Galilean transformation.

Souriau proposed a new approach, in which the *symplectic manifold of motions* of the mechanical system conceptually plays the central part.

The manifold of motions of a mechanical system (2)

It may seem strange to consider the set of all possible motions of a system, which is unknown as long as we have not determined all these possible motions. One may ask if it is really useful when we want to determine not all possible motions, but only one motion with prescribed initial data, since that motion is just one point of the (unknown) manifold of motion!

Souriau's answers to this objection are the following.

1. We know that the manifold of motions has a *symplectic structure*, and very often many things are known about its *symmetry properties*.

2. In classical (non-relativistic) mechanics, there exists a natural mathematical object which *does not depend on the choice of a particular reference frame* (even if the decriptions given to that object by different observers depend on the reference frame used by these observers). Souriau calls it the *evolution space* of the mechanical system.

The evolution space of a mechanical system

The knowledge of the equations which govern the system's evolution allows the full mathematical description of the *evolution space*, even when these equations are not yet solved.

Moreover, the symmetry properties of the *evolution space* are the same as those of the manifold of motions.

The *evolutuon space* of a classical mechanical system is an *odd-dimensional presymplectic manifold*. Its presymplectic form is called the *Lagrange form*. At each point of the evolution space, the *Lagrange form's kernel is* 1*-dimensional*. The evolution space is therefore endowed with a natural *isotropic foliation in curves*. Each of these curves describes a possible motion of the system. The *manifold of motions* is the set of these curves: it is the *quotient space* of the evolution space by its isotropic foliation. Its dimension is equal to *the dimension of the evolution space minus* 1. Of course that dimension is *even*.

Leibniz space-time of nonrelativistic Mechanics

In classical (non-relativistic) Mechanics, motions are described in *Leibniz space-time*. It is an *abstract 4-dimensional affine space $\mathcal{L}$* whose elements are called *events*. Once *units of time and of length* have been chosen, Leibniz space-time $\mathcal{L}$ is endowed with the following structure:

1. There exists an *affine submersion $\tau : \mathcal{L} \to \mathcal{T}$* of Leibniz space-time $\mathcal{L}$ onto an *oriented one-dimensional affine Euclidean space $\mathcal{T}$*. The space $\mathcal{T}$ is the *absolute time*, its orientation is *towards the future* and τ is the *date map*. To each event $z \in \mathcal{L}$ it associates the time $\tau(z)$ at which occured that event.

2. For each $t \in \mathcal{T}$, the *three-dimensional* affine subspace $\mathcal{E}_t = \tau^{-1}(t)$ of $\mathcal{L}$, called the *space at time t*, is endowed with an Euclidean metric.

3. Any *translation* of $\mathcal{L}$, when restricted to some three-dimensional subspace $\mathcal{E}_{t_1}$ (the space at time t_1), is an *isometry* of that Euclidean affine space onto another subspace $\mathcal{E}_{t_2}$ (the space at time t_2). Of course it may happen that $t_1 = t_2$.

Leibniz space-time and reference frames

The *abstract Galilean group* is the group of *orientation-preserving affine isomorphisms* of Leibniz space-time $\mathcal{L}$ which *preserve its structure*, *i.e.* the Euclidean structure on the one-dimensional space of absolute time $\mathcal{T}$, its orientation towards the future and, for each $t \in \mathcal{T}$, the Euclidean structure on $\mathcal{E}_t = \tau^{-1}(t)$, the space at time t. Of course it is a subgroup of the group of all affine isomorphisms of $\mathcal{L}$.

In Leibniz space-time $\mathcal{L}$ there exists a set $\mathcal{T}$ of absolute times (it can be identified with the set of all three-dimensional subspaces $\mathcal{E}_t \subset \mathcal{L}$, for all $t \in \mathcal{T}$) but *there is no absolute space*.

Observers, who do not have a direct access to Leibniz space-time $\mathcal{L}$, traditionally use *reference frames*. The reference frame of an observer O is a smooth diffeomorphism

$$\mathcal{R}_o : \mathcal{E}_o \times \mathcal{T}_o \to \mathcal{L}$$

with the following properties.

Leibniz space-time and reference frames (2)

1. $\mathcal{E}_o$ and $\mathcal{T}_o$ are oriented Euclidean *vector* spaces, respectively *three-dimensional* and *one-dimensional*, called the *observer's space* and the *observer's time*.

2. The map $\tau \circ \mathcal{R}_o : \mathcal{E}_o \times \mathcal{T}_o \to \mathcal{T}$, $(x, t) \mapsto \tau \circ \mathcal{R}_o(x, t)$, *does not depend on* $x \in \mathcal{E}_o$, and yields an Euclidean affine orientation preserving isometry $t \mapsto \tau \circ \mathcal{R}_o(\text{any } x, t)$ of the time of the observer, $\mathcal{T}_o$, onto the absolute time, $\mathcal{T}$.

3. For each fixed $t \in \mathcal{T}_o$, the partial map $x \mapsto \mathcal{R}_o(t, x)$ is an Euclidean affine isometry of the observer's space $\mathcal{E}_o$ onto the space at some fixed time $\mathcal{E}_{t_a}$, where $t_a = \tau \circ \mathcal{R}_o(\text{any } x, t)$.

For simplifying things we will assume that an *orientation* has been chosen on $\mathcal{L}$ which, together with the orientation of $\mathcal{T}$ towards the future, determines an orientation on $\mathcal{E}_t$, the space at time t, for all $t \in \mathcal{T}$, and we will consider only reference frames $\mathcal{R}_o$ for which the Euclidean isometries of $\mathcal{E}_o$ onto the spaces $\mathcal{E}_t$ at time t, for various $t \in \mathcal{T}$, are *positively oriented*.

Leibniz space-time and reference frames (3)

The *Galilean group* of an observer O whose space and time are, respectively, $\mathcal{E}_o$ and $\mathcal{T}_o$, is the group of affine maps of $\mathcal{E}_o \times \mathcal{T}_o$ onto itself which can be written as $(\overrightarrow{x}, t) \mapsto (\overrightarrow{x}', t')$, with

$$\begin{cases} \overrightarrow{x}' = g(\overrightarrow{x}) + t\,\overrightarrow{u} + \overrightarrow{w}\,, \\ t' = t + e\,, \end{cases}$$

where $g \in \mathrm{SO}(\mathcal{E}_o)$, $\overrightarrow{u}$ and $\overrightarrow{w} \in \mathcal{E}_o$ and $e \in \mathcal{T}_o$. The above equations can be conveniently written in matrix form

$$\begin{pmatrix} \overrightarrow{x}' \\ t' \\ 1 \end{pmatrix} = \begin{pmatrix} g & \overrightarrow{u} & \overrightarrow{w} \\ 0 & 1 & e \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \overrightarrow{x} \\ t \\ 1 \end{pmatrix}$$

The reference frame $\mathcal{R}_o : \mathcal{E}_o \times \mathcal{T}_o \to \mathcal{L}$ is said to be *inertial* or *Galilean* if, in addition to the conditions imposed to all reference frames, $\mathcal{R}_o$ is an *affine isomorphism*.

Leibniz space-time and reference frames (4)

A *Galilean reference frame* $\mathcal{R}_o$ determines a *group isomorphism* of the Galilean group of the observer O onto the abstract Galilean group, which associates to each element a in the Galilean group of O the Galilean transformation of $\mathcal{L}$

$$z \mapsto \mathcal{R}_o \circ a \circ \mathcal{R}_o^{-1}\,, \quad z \in \mathcal{L}\,.$$

The evolution space of a system of material points

Let us consider a finite number n of material points with masses m_i, $1 \leq i \leq n$. In the reference frame of some observer O, the motion of this system is described by a map $t \mapsto (\overrightarrow{x}_1(t), \ldots, \overrightarrow{x}_n(t))$ defined on some interval $I \subset \mathcal{T}_o$ of the time of the observer, with values in $\mathcal{E}_o^n$, the n-th power of the space of the observer. The equations of motion are

$$\begin{cases} \dfrac{\mathrm{d}\overrightarrow{x}_i(t)}{\mathrm{d}t} = \overrightarrow{v}_i(t)\,, \\ \dfrac{\mathrm{d}\overrightarrow{v}_i(t)}{\mathrm{d}t} = \overrightarrow{F}_i\bigl(t, \overrightarrow{x}(t), \overrightarrow{v}(t)\bigr)\,, \end{cases} \qquad 1 \leq i \leq n\,.$$

The vector $\overrightarrow{F}_i$ is the *total force* exerted on the i-th material point: it includes the effect of an external force field, the forces exerted by the other material points and, if the reference frame $\mathcal{R}_o$ is not Galilean, *centrifugal and Coriolis forces*. It is assumed to be a known function of the time, the positions and the velocities of the material points seen in the reference frame of the observer O.

The evolution space of a system of material points (2)

The *evolution space* of this system of material points, as seen by the observer O, is the set $\mathcal{V} = \mathcal{T}_o \times \mathcal{E}_o^n \times \mathcal{E}_o^n$, whose elements are $(t, \overrightarrow{x}_1, \ldots, \overrightarrow{x}_n, \overrightarrow{v}_1, \ldots, \overrightarrow{v}_n)$.

The *Lagrange form* on $\mathcal{V}$ is the differential 2-form

$$\sigma = \sum_{i=1}^{n} (m_i \mathrm{d}\overrightarrow{v}_i - \overrightarrow{F}_i \mathrm{d}t) \wedge (\mathrm{d}\overrightarrow{x}_i - \overrightarrow{v}_i \mathrm{d}t),$$

where summation over the product of components with the same index of the vector valued functions and differentials in an orthonormal basis of $\mathcal{E}_o$ is tacitely assumed.

At each point of $\mathcal{V}$ the rank of $\ker \sigma$ is equal to 1, and the equations of motion of the system written above simply mean that the vector

$$\left(1, \frac{\mathrm{d}\overrightarrow{x}_1}{\mathrm{d}t}, \ldots, \frac{\mathrm{d}\overrightarrow{x}_n}{\mathrm{d}t}, \frac{\mathrm{d}\overrightarrow{v}_1}{\mathrm{d}t}, \frac{\mathrm{d}\overrightarrow{v}_n}{\mathrm{d}t}\right)$$

lies in $\ker \sigma$.

The evolution space of a system of material points (3)

Taking the image by the reference frame $\mathcal{R}_o$ (and its natural prolongation to vectors and differential forms) of the evolution space $\mathcal{V}$ and of the Lagrange form σ, we can define an *intrinsic evolution space* $\mathcal{V}_a$ endowed with an *intrinsic Lagrange form* σ_a. Elements of the intrinsic evolution space are multiplets $(t, z_1, \ldots, z_n, \overrightarrow{V}_1, \overrightarrow{V}_n)$, where $t \in \mathcal{T}$ is an absolute time; for each i $(1 \leq i \leq n)$ $z_i \in \mathcal{E}_t$ is an element of the Leibniz space-time such that $\tau(x_i) = t$ and $\overrightarrow{V}_i$ is a 4-vector tangent to $\mathcal{L}$ at the point z_i whose projection on $\mathcal{T}$ (by the natural prolongation to vectors of the date map τ) is equal to 1.

One can check that the intrinsic evolution space and its intrinsic Lagrange form *do not depend* on the choice of the reference frame in which it was expressed, even when that reference frame is not Galilean.

More general evolution spaces

The evolution space and the Lagrange form can be determined for classical (non-relativistic) mechanical systems more general than those made by a finite number of material points, for example systems made by rigid bodies with ideal holonomic constraints, Lagrangian systems and Hamiltonian systems.

For a Hamiltonian system on a smooth symplectic manifold (M, ω) with a smooth (maybe time-dependent) Hamiltonian $H : M \times \mathbb{R} \to \mathbb{R}$, the *evolution space* is $M \times \mathbb{R}$ and the *Lagrange form* is the *Poincaré-Cartan 2-form*

$$\sigma = \omega - \mathrm{d}H \wedge \mathrm{d}t\,.$$

For a Lagrangian system with a smooth regular Lagrangian $L : TN \times \mathbb{R} \to \mathbb{R}$, the *evolution space* is $TN \times \mathbb{R}$ and the *Lagrange form* is

$$\sigma = \sum_{i=1}^{n} \mathrm{d}\left(\frac{\partial L(x,v,t)}{\partial v^i}\right) \wedge \mathrm{d}x^i - \mathrm{d}\left(\sum_{i=1}^{n} v^i \frac{\partial L(x,v,t)}{\partial v^i} - L(x,v,t)\right) \wedge \mathrm{d}t\,.$$

Closedness of the Lagrange form

We see that in the two above given examples (Hamiltonian systems and Lagrangian systems) the Lagrange form σ is *closed*, *i.e.* it satisfies $\mathrm{d}\sigma = 0$.

For a system made of a finite number n of material points, imposing to the Lagrange form to be closed restricts the the generality of the functions which express the total forces $\overrightarrow{F}_i$ ($1 \leq i \leq n$) exerted on the particles. However, σ is closed in many examples: electrically charged particles in an external electromagnetic field, gravitationally interacting material points, systems with ideal holonomic constraints, ... Souriau states the *closedness of the Lagrange form* σ as a *fundamental principle of Mechanics*; he calls it the *Maxwell Principle*.

The Maxwell Principle has an important consequence: the Lagrange form σ is a *presemplectic form* which projects on the quotient of the evolution space by its isotropic foliation, *i.e.* on the *manifold of motions*. This projected form is a *symplectic form*.

Symmetries of an isolated system

In classical (non-relativistic and non-quantum) *Analytical Mechanics*, an isolated mechanical system is made by a (maybe infinite) number of material points which evolve in the Leibniz space-time $\mathcal{L}$, interacting between them by *instantaneous interactions*. Let us consider, for example, the motion of a rigid body. It is assumed that the constraint forces between its different points ensure its *perfect rigidity* during its motion; they instantaneously take the values which ensure that rigidity. Moreover, the system being *isolated*, there is no external field of forces which could break the natural symmetries of the Leibniz space-time $\mathcal{L}$, since the forces exerted on a material point of the system all are created by other material points of that system.

Therefore, *the symmetries of the Leibniz space-time $\mathcal{L}$ all are too symmetries of any isolated mechanical system evolving in $\mathcal{L}$*.

In other words, *the group of symmetries of an isolated mechanical system contains the abstract Galilean group of symmetries of the Leibniz space-time $\mathcal{L}$*.

Momentum maps

For an observer O whose reference frame $\mathcal{R}_o$ is *Galilean*, the group of symmetries of the *evolution space* $\mathcal{V}$ of an isolated mechanical system, as it is seen in the reference frame $\mathcal{R}_o$, *contains the Galilean group of the observer*.

Therefore the Galilean group G_o of the observer acts on the presymplectic manifold $(\mathcal{V}, \sigma)$, the Lagrange form σ remaining invariant under that action. In many examples, that action is *Hamiltonian*: it means that there exists a *momentum map* $J : \mathcal{V} \to \mathcal{G}_o^*$, defined on the evolution space $\mathcal{V}$, taking its values in the dual space $\mathcal{G}_o^*$ of the Lie algebra $\mathcal{G}_o$ of the Galilean group G_o, which yields a Hamiltonian J_X

$$v \mapsto J_X(v) = \left\langle J(v), X \right\rangle, \quad v \in \mathcal{V}, \ X \in \mathcal{G}_o,$$

for the infinitesimal action on $\mathcal{V}$ of each infinitesimal symmetry $X \in \mathcal{G}_o$

Emmy Noether's theorem and equivariance of the momentum map

The Hamiltonian form of a theorem due to *Emmy Noether* states that the momentum map $J : \mathcal{V} \to \mathcal{G}_o^*$ keeps a constant value on each leaf of the isotropic foliation of the evolution space $(\mathcal{V}, \sigma)$. In other words, *each component of J* in a basis of $\mathcal{G}_o^*$ is a *first integral* of the equations of motion of the system.

Moreover, let us denote by $\Phi : G_o \times \mathcal{V} \to \mathcal{V}$ the action of the Galilean group G_o on the evolution space $\mathcal{V}$. There exists a unique *affine action* $a : G_o \times \mathcal{G}_o^* \to \mathcal{G}_o^*$ of the Galilean group G_o on the dual space $\mathcal{G}_o^*$ of its LIe algebra for which the momentum map J is *equivariant*, *i.e.* such that, for each $v \in \mathcal{V}$ and $g \in G_o$,

$$J\big(\Phi(g, v)\big) = a\big(g, J(v)\big) .$$

Symplectic cocycles

The affine action a is expressed as

$$a(g,\zeta) = \mathrm{Ad}^*_{g^{-1}}(\zeta) + \theta(g)\,, \quad g \in G_o,\ \zeta \in \mathcal{G}_o^*\,,$$

where $(g,\zeta) \mapsto \mathrm{Ad}^*_{g^{-1}}(\zeta)$ is the *coadjoint action* of the Galilean group G_o on the dual of its Lie algebra. We see that the *linear part* of the affine action a is the coadjoint action. The map $\theta : G_o \to \mathcal{G}_o^*$ is a *symplectic cocycle* of G_o for the coadjoint representation.

The momentum map $J : \mathcal{V} \to \mathcal{G}_o^*$ is determined only *up to addition of an arbitrary constant element in $\mathcal{G}_o^*$*. When the momentum map J is replaced by $J' = J + \mu$ (where $\mu \in \mathcal{G}_o^*$ is a constant), the cocycle θ is replaced by

$$\theta'(g) = \theta(g) + \left(\mu - \mathrm{Ad}^*_{g^{-1}}(\mu)\right).$$

We see that the additional term $g \mapsto \mu - \mathrm{Ad}^*_g \mu$ is a *symplectic coboundary* of the Galilean group G_o for the coadjoint action. Therefore the *symplectic cohomology class* of θ' is the same as that of θ, denoted by $[\theta]$: it is an *invariant* of the action Φ.

Action of the Galilean group on the manifold of motions

Since the action of any element of G_o on the evolution space $\mathcal{V}$ maps each isotropic leaf (*i.e* each *motion* of the system) onto another isotropic leaf (*i.e.* onto another *motion*), the Galilean group acts on the manifold of motions (M, σ) of the system.

When the action of G_o on $\mathcal{V}$ is Hamiltonian, its action on the symplectic manifold of motions (M, σ) too is Hamiltonian, and the symplectic cohomology class $[\theta]$ of that action remains the same: it is an *invariant* of the action of the Galilean group on the symplectic manifold of motions of the system.

We have seen that the components of the momentum map $J : \mathcal{V} \to \mathcal{G}_o^*$ are first integrals of the equations of motion. In many examples they have a clear physical meaning. The symplectic cohomology class $[\theta]$ of the action too has a physical meaning: as shown by Souriau, it can be interpreted as the *total mass* of the system.

Symplectic cocycles of the Galilean group

V. Bargmann (*Ann. Math.* **59**, 1954, pp 1–46) has proven that the symplectic cohomology space of the Galilean group is *one-dimensional*. Let us describe this space for the Galilean group G_o of the observer O (isomorphic to the abstract Galilean group which acts on Leibniz space-time $\mathcal{L}$). An element of G_o is a matrix

$$a = \begin{pmatrix} g & \overrightarrow{u} & \overrightarrow{w} \\ 0 & 1 & e \\ 0 & 0 & 1 \end{pmatrix}$$

where $g \in \mathrm{SO}(\mathcal{E}_o)$, $\overrightarrow{u}$ and $\overrightarrow{w} \in \mathcal{E}_o$ and $e \in \mathcal{T}_o$. An element of its Lie algebra is a matrix

$$X = \begin{pmatrix} \overrightarrow{\omega} & \overrightarrow{\beta} & \overrightarrow{\gamma} \\ 0 & 0 & \varepsilon \\ 0 & 0 & 0 \end{pmatrix}$$

where $\overrightarrow{\omega} \in \mathfrak{so}(\mathcal{E}_o)$, $\overrightarrow{\beta}$ and $\overrightarrow{\gamma} \in \mathcal{E}_o$ and $\varepsilon \in \mathcal{T}_o$.

Symplectic cocycles of the Galilean group (2)

Since the Euclidean vector space $\mathcal{E}_o$ is *three-dimensional and oriented*, the Lie algebra $\mathfrak{so}(\mathcal{E}_o)$ is identified with $\mathcal{E}_o$, an element $\overrightarrow{\omega} \in \mathfrak{so}(\mathcal{E})$ being identified with the vector $\overrightarrow{\omega} \in \mathcal{E}_o$, acting on $\mathcal{E}_o$ by the *vector product*

$$\overrightarrow{\omega} : \overrightarrow{x} \mapsto \overrightarrow{\omega} \times \overrightarrow{x} \quad \overrightarrow{x} \in \mathcal{E}_o\,.$$

The Lie algebra $\mathcal{G}_o$ can therefore be identified with $\mathcal{E}_o \times \mathcal{E}_o \times \mathcal{E}_o \times \mathcal{T}_o$, the element $X \in \mathcal{G}_o$ being identified with the *column vector*

$$X = \begin{pmatrix} \overrightarrow{\omega} & \overrightarrow{\beta} & \overrightarrow{\gamma} & \varepsilon \end{pmatrix}^T\,.$$

Using the *scalar product* in $\mathcal{E}_o$ and the product of line matrices with vector matrices as duality coupling, the dual space $\mathcal{G}_o^*$ of the Lie algebra $\mathcal{G}_o$ is identified with the space of *line vectors* of the form

$$\begin{pmatrix} \overrightarrow{I} & \overrightarrow{J} & \overrightarrow{K} & E \end{pmatrix}$$

with $\overrightarrow{I}$, $\overrightarrow{J}$ and $\overrightarrow{K} \in \mathcal{E}_o$ and $E \in \mathcal{T}_o$.

Symplectic cocycles of the Galilean group (3)

With the above stated identification, *all symplectic cocycles of the Galilean group are cohomologous to the cocycle*

$$\theta_m = m\theta_1$$

where $m \in \mathbb{R}$ is a real constant, and

$$\theta_1(a) = \begin{pmatrix} \overrightarrow{u} \times \overrightarrow{w} & \overrightarrow{w} - e\overrightarrow{u} & -\overrightarrow{u} & \frac{1}{2}\|\overrightarrow{u}\|^2 \end{pmatrix} \,.$$

I recall that

$$a = \begin{pmatrix} g & \overrightarrow{u} & \overrightarrow{w} \\ 0 & 1 & e \\ 0 & 0 & 1 \end{pmatrix} \,.$$

Example: the n body problem

We consider n material points which interact by gravitational attraction. The evolution space $\mathcal{V}$ of the system, as seen by the observer O, is the set of all

$$(\overrightarrow{x}_1, \dots, \overrightarrow{x}_n, \overrightarrow{v}_1, \dots, \overrightarrow{v}_n, t) \in {\mathcal{E}_o}^{2n} \times \mathcal{T}_o \quad \text{with } \overrightarrow{x}_i \neq \overrightarrow{x}_j \text{ if } i \neq j\,.$$

The gravitational forces are described by a *potential energy* $U(\overrightarrow{x}_1, \dots, \overrightarrow{x}_n)$ which only depends on the mutual distances $\|\overrightarrow{x}_i - \overrightarrow{x}_j\|$ between all pairs of the n material points. The momentum map $J : \mathcal{V} \to \mathcal{G}_o^*$ is

$$J(\overrightarrow{x}, \overrightarrow{v}, t) = \begin{pmatrix} A & B & C & D \end{pmatrix},$$

with

$$A = \sum_{i=1}^n m_i \overrightarrow{x}_i \times \overrightarrow{v}_i \qquad B = -\sum_{i=1}^n m_i(\overrightarrow{x}_i - t\overrightarrow{v}_i)$$

$$C = \sum_{i=1}^n m_i \overrightarrow{v}_i \qquad D = -\left(\frac{1}{2}\sum_{i=1}^n m_i\|\overrightarrow{v}_i\|^2 + U(\overrightarrow{x})\right).$$

Example: the n body problem (2)

We see that A is the *total angular momentum* with respect to the origin, C the *total linear momentum* and D the opposite of the *total energy* (kinetic plus potential) of the system. When $\displaystyle\sum_{i=1}^{n} m_i \neq 0$, we can write

$$\frac{B}{\displaystyle\sum_{i=1}^{n} m_i} = -\overrightarrow{x}_G + t\,\frac{C}{\displaystyle\sum_{i=1}^{n} m_i}\,, \quad \text{with } \overrightarrow{x}_G = \frac{\displaystyle\sum_{i=1}^{n} m_i \overrightarrow{x}_i}{\displaystyle\sum_{i=1}^{n} m_i}\,.$$

$B =$ Constant means that $\overrightarrow{x}_G$ is an affine function of time: the center of mass moves on a straight line at a constant velocity. The cocycle for which the momentum map is equivariant is

$$\theta = \left(\sum_{i=1}^{n} m_i\right) \theta_1\,.$$

Other applications in classical Mechanics

More generally, for classical isolated conservative mechanical systems made of material points and rigid bodies which may interact by perfect holonomic constraints, one can prove that the symplectic cocycle for which the momentum map is equivariant is $m\theta_1$, m being the *total mass* of the system. Of course for usual mechanical systems $m > 0$, but it is possible to consider the manifold of motions of systems with $m = 0$ or even with $m < 0$.

The subset of elements of the Galilean group G_o which are expressed as

$$\begin{pmatrix} \mathrm{id}_{\mathcal{E}_o} & \overrightarrow{u} & \overrightarrow{w} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is an Abelian six-dimensional normal subgroup $\widetilde{G}_o$ of G_o isomorphic to $\mathcal{E}_o \times \mathcal{E}_o$. The quotient group $G_o/\widetilde{G}_o$ is isomorphic to $\mathrm{SO}(\mathcal{E}_o) \times \mathcal{T}_o$.

Other applications in classical Mechanics (2)

The manifold of motions of of an isolated classical mechanical system of total mass $m \neq 0$ can be split into a product $M = M_1 \times M_2$. The factor M_1, isomorphic to a six-dimensional vector space, is the *manifold of motions of the center of mass*, and the factor M_2 the *manifold of motions around the center of mass* (in which the center of mass remains at rest at the origin of the space $\mathcal{E}_o$). The Galilean group acts separately on M_1 and on M_2, its action on M_2 occuring through its projection onto the quotient group $G_o/\widetilde{G}_o \equiv \mathrm{SO}(\mathcal{E}_o) \times \mathcal{T}_o$. This property is called the *barycentric decomposition*.

Relativistic Mechanics

In Special Relativity, the Leibniz space-time $\mathcal{L}$ is no longer the frame in which is described the motion of material bodies. It is replaced by the *Minkowski space-time* $\mathcal{M}$, whose symmetry group is the Poincaré group. What should be considered as the evolution space of a system in Special Relativity is a difficult question, even when the system contains only a finite number of interacting point particles, because their interactions occur by means of fields created by the particles themselves, which propagate at a finite velocity and carry away a part of the particle's energy.

However, it seems natural to state as a postulate that the *manifold of motions of an isolated system* in Special Relativity is a *symplectic manifold* on which the Poincaré group acts by a symplectic action. Since the Lie algebra of the Poincaré group is equal to its derived ideal, a symplectic action is automatically Hamiltonian. So there always exists a momentum map defined on the manifold of motions with values in the dual of the Lie algebra of the Poincaré group.

Relativistic Mechanics (2)

Contrary to that of the Galilean group, the symplectic cohomology of the Poincaré group is trivial. Therefore the momentum map can always be chosen to be equivariant with respect to the coadjoint action of the Poincaré group on the dual of its Lie algebra. There is no more a cocycle corresponding to the total mass of the system.

Elementary mechanical systems

An *elementary system* is a system such that when it is isolated, the symmetry group of space-time (the Galilean group if the system is considered in classical mechanics, or the Poincaré group if it is considered in Special Relativity) acts *transitively* on its manifold of motions. The manifold of motions is therefore a *symplectic homogeneous space* of the symmetry group. The symplectic homogeneous spaces of the Poincaré group are the *coadjoint orbits*. Their classification has many features of the classification of elementary particles, by their mass and spin.

For the Galilean group things are slightly more complicated because one must account for the cocycle: homogeneous symplectic spaces are orbits of an affine action of the Galilean group on the dual of its Lie algebra, and the symplectic form on these orbits has an additional term involving that cocycle.

Many thanks to Gabriel Catren and to all the members of the organizing committee for inviting me to present a talk at this workshop.

And thanks to all the persons who patiently listened to my talk!

References

The reader will find below the references to the works of Lagrange and Poisson discussed in my talk, and to the book and paper of Jean-Marie Souriau which contain almost all the matters I have presented.

He will find more about the works of Poisson in the recent book "Siméon-Denis Poisson, les mathématiques au service de la science" and a very thorough discussion of the Noether theorems in the beautiful book "The Noether theorems" by Y. Kosmann-Schwarzbach.

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