

Actions of Lie groups and Lie algebras on symplectic and Poisson manifolds. Application to Lagrangian and Hamiltonian systems

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1. Introduction

I present in this talk some tools in *Symplectic and Poisson Geometry* in view of their applications in *Geometric Mechanics* and *Mathematical Physics*.

In parts 2 and 3 I discuss the *Lagrangian formalism* and *Lagrangian symmetries*, and in parts 4 and 5 the *Hamiltonian formalism* and *Hamiltonian symmetries*. The *Tulczyjew isomorphisms*, which explain some aspects of the relations between the Lagrangian and Hamiltonian formalisms, are presented at the end of part 4.

Part 6 discusses Jean-Marie Souriau's theory of *Thermodynamics on Lie groups*.

Finally, the *Euler-Poincaré* equation is presented in an Appendix.

2. The Lagrangian formalism

The principles of Mechanics were stated by the great English mathematician *Isaac Newton* (1642–1727) in his book *Philosophia Naturalis Principia Mathematica* published in 1687 [27].

On this basis, a little more than a century later, *Joseph Louis Lagrange* (1736–1813) in his book *Mécanique analytique* [16] derived the equations (today known as the *Euler-Lagrange equations*) which govern the motion of a mechanical system made of any number of material points or rigid material bodies, eventually submitted to external forces, interacting between themselves by very general forces.

The *configuration space* and the *space of kinematic states* of the system are, respectively, a smooth n -dimensional manifold N and its tangent bundle TN , which is $2n$ -dimensional. In local coordinates a configuration of the system is determined by the n coordinates x^1, \dots, x^n of a point in N , and a kinematic state by the $2n$ coordinates $x^1, \dots, x^n, v^1, \dots, v^n$ of a vector tangent to N at some point in N .

2. The Lagrangian formalism

2.1. The Euler-Lagrange equations

When the mechanical system is *conservative*, the Euler-Lagrange equations involve a single real valued function L called the *Lagrangian* of the system, defined on the product of the real line \mathbb{R} (spanned by the variable t representing the time) with the manifold TN of *kinematic states* of the system. In local coordinates, the Lagrangian L is expressed as a function of the $2n + 1$ variables, $t, x^1, \dots, x^n, v^1, \dots, v^n$ and the Euler-Lagrange equations have the remarkably simple form

$$\frac{d}{dt} \left(\frac{\partial L}{\partial v^i}(t, x(t), v(t)) \right) - \frac{\partial L}{\partial x^i}(t, x(t), v(t)) = 0, \quad 1 \leq i \leq n,$$

where $x(t)$ stands for $x^1(t), \dots, x^n(t)$ and $v(t)$ for $v^1(t), \dots, v^n(t)$ with, of course,

$$v^i(t) = \frac{dx^i(t)}{dt}, \quad 1 \leq i \leq n.$$

2. The Lagrangian formalism

2.2. Hamilton's principle of stationary action

The great Irish mathematician *William Rowan Hamilton* (1805–1865) observed [8, 9] that the Euler-Lagrange equations can be obtained by applying the standard techniques of *Calculus of Variations*, due to *Leonhard Euler* (1707–1783) and *Joseph Louis Lagrange*, to the *action integral*

$$I_L(\gamma) = \int_{t_0}^{t_1} L\left(t, x(t), v(t) = \frac{dx(t)}{dt}\right) dt,$$

where $\gamma : [t_0, t_1] \rightarrow N$ is a smooth curve in N parametrized by the time t . These equations express the fact that the action integral $I_L(\gamma)$ is *stationary* with respect to any smooth infinitesimal variation of γ with fixed end-points $(t_0, \gamma(t_0))$ and $(t_1, \gamma(t_1))$. This fact is today called *Hamilton's principle of stationary action*.

This principle does not appear explicitly in Lagrange's book in which the Euler-Lagrange equations are obtained by a very clever evaluation of the virtual work of inertial forces for a smooth infinitesimal variation of the motion.

2. The Lagrangian formalism

2.3. The Euler-Cartan theorem

The *Lagrangian formalism* is the use of Hamilton's principle of stationary action for the derivation of the equations of motion of a system. It is widely used in Mathematical Physics, often with more general Lagrangians involving more than one independent variable and higher order partial derivatives of dependent variables. For simplicity I will consider here only the Lagrangians of (maybe time dependent) conservative mechanical systems.

An intrinsic geometric expression of the Euler-Lagrange equations, which does not use local coordinates, was obtained by the great French mathematician *Élie Cartan* (1869–1951). Let T^*N be the cotangent space to the configuration manifold N (often called the *phase space* of the mechanical system), θ_N be its Liouville 1-form, $\mathcal{L}_L = d_{\text{vert}}L : \mathbb{R} \times TN \rightarrow T^*N$ be the Legendre map and $E : \mathbb{R} \times TN \rightarrow \mathbb{R}$ be the *energy function*

$$E_L(t, v) = \langle d_{\text{vert}}L(t, v), v \rangle - L(t, v), \quad v \in TN.$$

The Lagrangian formalism (5)

2.3. The Euler-Cartan theorem (2)

The 1-form on $\mathbb{R} \times TN$

$$\widehat{\omega}_L = \mathcal{L}_L^* \theta_N - E_L(t, v) dt$$

is called the *Euler-Poincaré* 1-form. The *Euler-Cartan theorem*, due to Élie Cartan, asserts that the action integral $I_L(\gamma)$ is stationary at a smooth parametrized curve $\gamma : [t_0, t_1] \rightarrow N$, with respect to smooth infinitesimal variations of γ with fixed end-points, if and only if

$$i \left(\frac{d}{dt} \left(t, \frac{d\gamma(t)}{dt} \right) \right) d\widehat{\omega}_L \left(t, \frac{d\gamma(t)}{dt} \right) = 0.$$

In his beautiful book [?], *Jean-Marie Souriau* uses a slightly different terminology: for him the odd-dimensional space $\mathbb{R} \times TN$ is the *evolution space* of the system, and the exact 2-form $d\widehat{\omega}_L$ on that space is the *Lagrange form*. He defines that 2-form in a setting more general than that of the Lagrangian formalism.

3. Lagrangian symmetries

Let N be the configuration space of a conservative Lagrangian mechanical system with a smooth Lagrangian, maybe time dependent, $L : \mathbb{R} \times TN \rightarrow \mathbb{R}$. Let $\widehat{\omega}_L$ be the Poincaré-Cartan 1-form on the evolution space $\mathbb{R} \times TN$. Several kinds of symmetries can be defined, which very often are special cases of *infinitesimal symmetries of the Poincaré-Cartan form*, which play an important part in the famous *Noether theorem*.

3.1. Infinitesimal symmetries of the Poincaré-Cartan form

Definition

An *infinitesimal symmetry* of the Poincaré-Cartan form $\widehat{\omega}_L$ is a vector field Z on $\mathbb{R} \times TN$ such that

$$\mathcal{L}(Z)\widehat{\omega}_L = 0,$$

$\mathcal{L}(Z)$ denoting the Lie derivative of differential forms with respect to Z .

3. Lagrangian symmetries

3.1. Infinitesimal symmetries of the Poincaré-Cartan form (2)

Examples

- 1 Let us assume that the Lagrangian L does not depend on the time $t \in \mathbb{R}$, *i.e.* is a smooth function on TN . The vector field on $\mathbb{R} \times TN$ denoted by $\frac{\partial}{\partial t}$, whose projection on \mathbb{R} is equal to 1 and whose projection on TN is 0, is an infinitesimal symmetry of $\widehat{\omega}_L$.
- 2 Let X be a smooth vector field on N and \overline{X} be its canonical lift to the tangent bundle TN . We still assume that L does not depend on the time t . Moreover we assume that \overline{X} is an infinitesimal symmetry of the Lagrangian L , *i.e.* that $\mathcal{L}(\overline{X})L = 0$. Considered as a vector field on $\mathbb{R} \times TN$ whose projection on the factor \mathbb{R} is 0, \overline{X} is an infinitesimal symmetry of $\widehat{\omega}_L$.

3. Lagrangian symmetries

3.2. The Noether theorem in Lagrangian formalism

Theorem (E. Noether's theorem in Lagrangian formalism)

Let Z be an infinitesimal symmetry of the Poincaré-Cartan form $\widehat{\omega}_L$. For each possible motion $\gamma : [t_0, t_1] \rightarrow N$ of the Lagrangian system, the function, defined on $\mathbb{R} \times TN$,

$$i(Z)\widehat{\omega}_L$$

keeps a constant value along the parametrized curve

$$t \mapsto \left(t, \frac{d\gamma(t)}{dt} \right).$$

Example

When the Lagrangian L does not depend on time, application of Emmy Noether's theorem to the vector field $\frac{\partial}{\partial t}$ shows that the energy E_L remains constant during any possible motion of the system, since $i\left(\frac{\partial}{\partial t}\right)\widehat{\omega}_L = -E_L$.

3. Lagrangian symmetries

3.2. The Noether theorem in Lagrangian formalism (2)

Remark

There exists many generalizations of the Noether theorem. For example, if instead of being an infinitesimal symmetry of $\widehat{\omega}_L$, i.e. instead of satisfying

$$\mathcal{L}(Z)\widehat{\omega}_L = 0$$

the vector field Z satisfies

$$\mathcal{L}(Z)\widehat{\omega}_L = df,$$

where $f : \mathbb{R} \times TM \rightarrow \mathbb{R}$ is a smooth function, which implies of course

$$\mathcal{L}(Z)(d\widehat{\omega}_L) = 0,$$

the function

$$i(Z)\widehat{\omega}_L - f$$

keeps a constant value along $t \mapsto \left(t, \frac{d\gamma(t)}{dt}\right)$.

3. Lagrangian symmetries

3.3. The Lagrangian momentum map

The Lie bracket of two infinitesimal symmetries of $\widehat{\omega}_L$ is also an infinitesimal symmetry of $\widehat{\omega}_L$. Let us therefore assume that there exists a finite dimensional Lie algebra of vector fields on $\mathbb{R} \times TN$ whose elements are infinitesimal symmetries of $\widehat{\omega}_L$.

Definition

Let $\psi : \mathcal{G} \rightarrow A^1(\mathbb{R} \times TN)$ be a Lie algebras homomorphism of a finite-dimensional real Lie algebra \mathcal{G} into the Lie algebra of smooth vector fields on $\mathbb{R} \times TN$ such that, for each $X \in \mathcal{G}$, $\psi(X)$ is an infinitesimal symmetry of $\widehat{\omega}_L$. The Lie algebras homomorphism ψ is said to be a *Lie algebra action on $\mathbb{R} \times TN$ by infinitesimal symmetries of $\widehat{\omega}_L$* . The map $K_L : \mathbb{R} \times TN \rightarrow \mathcal{G}^*$, which takes its values in the dual \mathcal{G}^* of the Lie algebra \mathcal{G} , defined by

$$\langle K_L(t, v), X \rangle = i(\psi(X))\widehat{\omega}_L(t, v), \quad (t, v) \in \mathbb{R} \times TN,$$

is called the *Lagrangian momentum* of the Lie algebra action ψ .

3. Lagrangian symmetries

3.3. The Lagrangian momentum map (2)

Corollary (of E. Noether's theorem)

Let $\psi : \mathcal{G} \rightarrow A^1(\mathbb{R} \times TM)$ be an action of a finite-dimensional real Lie algebra \mathcal{G} on the evolution space $\mathbb{R} \times TN$ of a conservative Lagrangian system, by infinitesimal symmetries of the Poincaré-Cartan form $\widehat{\omega}_L$. For each possible motion $\gamma : [t_0, t_1] \rightarrow N$ of that system, the Lagrangian momentum map K_L keeps a constant value along the parametrized curve $t \mapsto \left(t, \frac{d\gamma(t)}{dt} \right)$.

3. Lagrangian symmetries

3.3. The Lagrangian momentum map (3)

Example

Let us assume that the Lagrangian L does not depend explicitly on the time t and is invariant by the canonical lift to the tangent bundle of the action on N of the six-dimensional group of Euclidean displacements (rotations and translations) of the physical space. The corresponding infinitesimal action of the Lie algebra of infinitesimal Euclidean displacements (considered as an action on $\mathbb{R} \times TN$, the action on the factor \mathbb{R} being trivial) is an action by infinitesimal symmetries of $\widehat{\omega}_L$. The *six components of the Lagrangian momentum map* are the *three components of the total linear momentum* and the *three components of the total angular momentum*.

Remark

These results are valid *without any assumption of hyper-regularity* of the Lagrangian.

4. The Hamiltonian formalism

The Lagrangian formalism can be applied to any smooth Lagrangian. Its application yields *second order differential equations* on $\mathbb{R} \times TN$ (in local coordinates, the *Euler-Lagrange equations*) which in general *are not solved with respect to the second order derivatives of the unknown functions with respect to time*. The classical existence and unicity theorems for the solutions of differential equations (such as the *Cauchy-Lipschitz theorem*) therefore *cannot be applied to these equations*.

Under the additional assumption that the Lagrangian is *hyper-regular*, a very clever change of variables discovered by *William Rowan Hamilton*¹ allows a new formulation of these equations in the framework of *symplectic geometry*. The *Hamiltonian formalism* is the use of these new equations. It was later generalized independently of the Lagrangian formalism.

¹*Lagrange* obtained however Hamilton's equations before *Hamilton*, but only in a special case, for the slow "variations of constants" such as the orbital parameters of planets in the solar system.

4. The Hamiltonian formalism

4.1. Hyper-regular Lagrangians

Assume that for each fixed value of the time $t \in \mathbb{R}$, the map $v \mapsto \mathcal{L}_L(t, v)$ is a smooth diffeomorphism of the tangent bundle TN onto the cotangent bundle T^*N . Equivalent assumption: the map $(\text{id}_{\mathbb{R}}, \mathcal{L}_L) : (t, v) \mapsto (t, \mathcal{L}_L(t, v))$ is a smooth diffeomorphism of $\mathbb{R} \times TN$ onto $\mathbb{R} \times T^*N$. The Lagrangian L is then said to be *hyper-regular*. The equations of motion can be written on $\mathbb{R} \times T^*N$ instead of $\mathbb{R} \times TN$.

Let $H_L : \mathbb{R} \times T^*N \rightarrow \mathbb{R}$ be the function, called the *Hamiltonian* associated to the Lagrangian L ,

$$H_L(t, p) = E_L \circ (\text{id}_{\mathbb{R}}, \mathcal{L}_L)^{-1}(t, p), \quad t \in \mathbb{R}, \quad p \in T^*N,$$

$E_L : \mathbb{R} \times TN \rightarrow \mathbb{R}$ being the *energy function*. The *Poincaré-Cartan 1-form* $\widehat{\omega}_L$ on $\mathbb{R} \times TN$ is the pull-back, by the diffeomorphism $(\text{id}_{\mathbb{R}}, \mathcal{L}_L) : \mathbb{R} \times TN \rightarrow \mathbb{R} \times T^*N$, of the 1-form on $\mathbb{R} \times T^*N$

$$\widehat{\omega}_H = \theta_N - H dt,$$

where θ_N is the *Liouville 1-form* on T^*N .

4. The Hamiltonian formalism

4.2. Presymplectic manifolds

The 1-form $\widehat{\omega}_{H_L}$ on $\mathbb{R} \times T^*N$ is called the *Poincaré-Cartan 1-form in Hamiltonian formalism*. It is related to the Poincaré-Cartan 1-form $\widehat{\omega}_L$ on $\mathbb{R} \times TN$, called the *Poincaré-Cartan 1-form in Lagrangian formalism*, by

$$\widehat{\omega}_L = (\text{id}_{\mathbb{R}}, \mathcal{L}_L)^* \widehat{\omega}_{H_L}.$$

The exterior derivatives $d\widehat{\omega}_L$ and $d\widehat{\omega}_{H_L}$ of the Poincaré-Cartan 1-forms in the Lagrangian and Hamiltonian formalisms both are *presymplectic 2-forms* on the odd-dimensional manifolds $\mathbb{R} \times TN$ and $\mathbb{R} \times T^*N$, respectively. At any point of these manifolds, the kernels of these closed 2 forms are 1-dimensional, therefore determine a *foliation into smooth curves* of these manifolds. The Euler-Cartan theorem shows that each of these curves is a possible *motion* of the system, described either in the Lagrangian formalism, or in the Hamiltonian formalism, respectively.

4. The Hamiltonian formalism

4.2. Presymplectic manifolds (2)

The set of all possible motions of the system, called by Jean-Marie Souriau the *manifold of motions* of the system, is described in the Lagrangian formalism by the quotient of the Lagrangian evolution space $\mathbb{R} \times TM$ by its foliation into curves determined by $\ker d\widehat{\omega}_L$, and in the Hamiltonian formalism by the quotient of the Hamiltonian evolution space $\mathbb{R} \times T^*M$ by its foliation into curves determined by $\ker d\widehat{\omega}_{H_L}$. Both are (maybe non-Hausdorff) *symplectic manifolds*, the projections on these quotient manifolds of the presymplectic forms $d\widehat{\omega}_L$ and $d\widehat{\omega}_h$ both being symplectic forms. Of course the diffeomorphism $(\text{id}_{\mathbb{R}}, \mathcal{L}_L) : \mathbb{R} \times TN \rightarrow \mathbb{R} \times T^*N$ projects onto a symplectomorphism between the Lagrangian and Hamiltonian descriptions of the manifold of motions of the system.

4. The Hamiltonian formalism

4.3. The Hamilton equation

Let $\psi : [t_0, t_1] \rightarrow T^*N$ be the map

$$\psi(t) = \mathcal{L}_L \left(t, \frac{d\gamma(t)}{dt} \right).$$

Since $d\widehat{\omega}_H = d\theta_N - dH_L \wedge dt$, the parametrized curve $t \mapsto \gamma(t)$ is a motion of the system if and only if the parametrized curve $t \mapsto \psi(t)$ satisfies both

$$\begin{cases} i \left(\frac{d\psi(t)}{dt} \right) d\theta_N = -dH_{L_t}, \\ \frac{d}{dt} (H_L(t, \psi(t))) = \frac{\partial H_L}{\partial t}(t, \psi(t)), \end{cases}$$

where $dH_{L_t} = dH_L - \frac{\partial H_L}{\partial t} dt$ is the differential of the function $H_{L_t} : T^*N \rightarrow \mathbb{R}$ in which the time t is considered as a parameter with respect to which there is no differentiation.

4. The Hamiltonian formalism

4.3. The Hamilton equation (2)

The first equation

$$i\left(\frac{d\psi(t)}{dt}\right) d\theta_N = -dH_{L_t}$$

is the *Hamilton equation*. In local coordinates $x^1, \dots, x^n, p_1, \dots, p_n$ on T^*N associated to the local coordinates x^1, \dots, x^n on N , it is expressed as

$$\begin{cases} \frac{dx^i(t)}{dt} = \frac{\partial H_L(t, x, p)}{\partial p_i}, \\ \frac{dp_i(t)}{dt} = -\frac{\partial H_L(t, x, p)}{\partial x^i}, \end{cases} \quad 1 \leq i \leq n.$$

The second equation

$$\frac{d}{dt}\left(H_L(t, \psi(t))\right) = \frac{\partial H_L}{\partial t}(t, \psi(t))$$

is the *energy equation*. It is automatically satisfied when the Hamilton equation is satisfied.

4. The Hamiltonian formalism

4.3. The Hamilton equation (3)

The 2-form $d\theta_N$ is a symplectic form on the cotangent bundle T^*N , called its *canonical symplectic form*. We have shown that when the Lagrangian L is hyper-regular, the equations of motion can be written in *three equivalent manners*:

- 1 as the *Euler-Lagrange equations* on $\mathbb{R} \times TM$,
- 2 as the equations given by the *kernels of the presymplectic forms* $d\hat{\omega}_L$ or $d\hat{\omega}_{H_L}$ which determine the foliations into curves of the evolution spaces $\mathbb{R} \times TM$ in the Lagrangian formalism, or $\mathbb{R} \times T^*M$ in the Hamiltonian formalism,
- 3 as the *Hamilton equation* associated to the Hamiltonian H_L on the symplectic manifold $(T^*N, d\theta_N)$, often called the *phase space* of the system.

4. The Hamiltonian formalism

4.4. The Tulczyjew isomorphisms

Around 1974, *W.M. Tulczyjew* [35, 36] discovered ² two remarkable vector bundles isomorphisms $\alpha_N : TT^*N \rightarrow T^*TN$ and $\beta_N : TT^*N \rightarrow T^*T^*N$.

The first one α_N is an isomorphism of the bundle $(TT^*N, T\pi_N, TN)$ onto the bundle (T^*TN, π_{TN}, TN) , while the second β_N is an isomorphism of the bundle $(TT^*N, \tau_{T^*N}, T^*N)$ onto the bundle $(T^*T^*N, \pi_{T^*N}, T^*N)$.

$$\begin{array}{ccccc} T^*T^*N & \xleftarrow{\beta_N} & TT^*N & \xrightarrow{\alpha_N} & T^*TN \\ \pi_{T^*N} \downarrow & \swarrow \tau_{T^*N} & & \searrow T\pi_N & \downarrow \pi_{TN} \\ T^*N & & & & TN \\ & \searrow \pi_N & & \swarrow \tau_N & \\ & & N & & \end{array}$$

² β_N was probably known long before 1974, but I believe that α_N , much more hidden, was noticed by Tulczyjew for the first time.

4. The Hamiltonian formalism

4.4. The Tulczyjew isomorphisms (2)

Since they are the total spaces of cotangent bundles, the manifolds T^*TN and T^*T^*N are endowed with the Liouville 1-forms θ_{TN} and θ_{T^*N} , and with the canonical symplectic forms $d\theta_{TN}$ and $d\theta_{T^*N}$, respectively.

Using the isomorphisms α_N and β_N , we can therefore define on TT^*N two 1-forms $\alpha_N^*\theta_{TN}$ and $\beta_N^*\theta_{T^*N}$, and two symplectic 2-forms $\alpha_N^*(d\theta_{TN})$ and $\beta_N^*(d\theta_{T^*N})$.

The very remarkable property of the isomorphisms α_N and β_N is that *the two symplectic forms so obtained on TT^*N are equal!*

$$\alpha_N^*(d\theta_{TN}) = \beta_N^*(d\theta_{T^*N}).$$

The 1-forms $\alpha_N^*\theta_{TN}$ and $\beta_N^*\theta_{T^*N}$ are not equal, their difference is the differential of a smooth function.

4. The Hamiltonian formalism

4.4. The Tulczyjew isomorphisms (3)

Let $L : TN \rightarrow \mathbb{R}$ and $H : T^* \rightarrow \mathbb{R}$ be two smooth real valued functions, defined on TN and on T^*N , respectively.

The graphs $dL(TN)$ and $dH(T^*N)$ of their differentials are Lagrangian submanifolds of the symplectic manifolds $(T^*TN, d\theta_{TN})$ and $(T^*T^*N, d\theta_{T^*N})$.

Their pull-backs $\alpha_N^{-1}(dL(TN))$ and $\beta_N^{-1}(dH(T^*N))$ by the symplectomorphisms α_N and β_N are therefore two Lagrangian submanifolds of the manifold TT^*N endowed with the symplectic form $\alpha_N^*(d\theta_{TN})$, which is equal to the symplectic form $\beta_N^*(d\theta_{T^*N})$.

The following theorem enlightens some aspects of the relationships between the Hamiltonian and the Lagrangian formalisms.

4. The Hamiltonian formalism

4.4. The Tulczyjew isomorphisms (4)

Theorem (W.M. Tulczyjew)

Let $X_H : T^*N \rightarrow TT^*N$ the Hamiltonian vector field on the symplectic manifold $(T^*N, d\theta_N)$ associated to the Hamiltonian $H : T^*N \rightarrow \mathbb{R}$, defined by $i(X_H)d\theta_N = -dH$. Then

$$X_H(T^*N) = \beta_N^{-1}(dH(T^*N)).$$

Moreover, the equality

$$\alpha_N^{-1}(dL(TN)) = \beta_N^{-1}(dH(T^*N))$$

if and only if the Lagrangian L is hyper-regular and such that

$$dH = d(E_L \circ \mathcal{L}_L^{-1}),$$

where $\mathcal{L}_L : TN \rightarrow T^*N$ is the Legendre map and $E_L : TN \rightarrow \mathbb{R}$ the energy associated to the Lagrangian L .

4. The Hamiltonian formalism

4.4. The Tulczyjew isomorphisms (5)

When L is not hyper-regular, $\alpha_N^{-1}(\mathrm{d}L(TN))$ still is a Lagrangian submanifold of the symplectic manifold $(TT^*N, \alpha_N^*(\mathrm{d}\theta_{TN}))$, but it is no more the graph of a smooth vector field X_H defined on T^*N . Tulczyjew proposes to consider this Lagrangian submanifold as an *implicit Hamilton equation* on T^*N .

These results can be extended to Lagrangians and Hamiltonians which may depend on time.

4. The Hamiltonian formalism

4.5. The Hamiltonian formalism on symplectic manifolds

In pure mathematics as well as in applications of mathematics to Mechanics and Physics, symplectic manifolds other than cotangent bundles are encountered. A theorem due to the french mathematician *Gaston Darboux* (1842–1917) asserts that any symplectic manifold (M, ω) is of even dimension $2n$ and is locally isomorphic to the cotangent bundle to a n -dimensional manifold: in a neighbourhood of each of its point there exist local coordinates $(x^1, \dots, x^n, p_1, \dots, p_n)$ with which the symplectic form ω is expressed exactly as the canonical symplectic form of a cotangent bundle:

$$\omega = \sum_{i=1}^n dp_i \wedge dx^i.$$

4. The Hamiltonian formalism

4.5. The Hamiltonian formalism on symplectic manifolds (2)

Let (M, ω) be a symplectic manifold and $H : \mathbb{R} \times M \rightarrow \mathbb{R}$ a smooth function, said to be a *time-dependent Hamiltonian*. It determines a *time-dependent Hamiltonian vector field* X_H on M , such that

$$i(X_H)\omega = -dH_t,$$

$H_t : M \rightarrow \mathbb{R}$ being the function H in which the variable t is considered as a parameter with respect to which no differentiation is made.

The Hamilton equation determined by H is the differential equation

$$\frac{d\psi(t)}{dt} = X_H(t, \psi(t)).$$

The Hamiltonian formalism can therefore be applied to any smooth, maybe time dependent Hamiltonian on M , even when there is no associated Lagrangian.

4. The Hamiltonian formalism

4.6. The Hamiltonian formalism on Poisson manifolds

The Hamiltonian formalism is not limited to symplectic manifolds: it can be applied, for example, to *Poisson manifolds* [20].

Definition

A *Poisson manifold* is a smooth manifold P whose algebra of smooth functions $C^\infty(P, \mathbb{R})$ is endowed with a bilinear composition law, called the *Poisson bracket*, which associates to any pair (f, g) of smooth functions on P another smooth function denoted by $\{f, g\}$, that composition satisfying the three properties

- 1 it is skew-symmetric,

$$\{g, f\} = -\{f, g\},$$

- 2 it satisfies the *Jacobi identity*

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0,$$

- 3 it satisfies the Leibniz identity

$$\{f, gh\} = \{f, g\}h + g\{f, h\}.$$

4. The Hamiltonian formalism

4.6. The Hamiltonian formalism on Poisson manifolds (2)

On a Poisson manifold P , the Poisson bracket $\{f, g\}$ of two smooth functions f and g can be expressed by means of a smooth field of bivectors Λ :

$$\{f, g\} = \Lambda(df, dg), \quad f \text{ and } g \in C^\infty(P, \mathbb{R}),$$

called the *Poisson bivector field* of P . The considered Poisson manifold is denoted by (P, Λ) . The Poisson bivector field Λ satisfies

$$[\Lambda, \Lambda] = 0,$$

where the bracket $[,]$ in the left hand side is the *Schouten-Nijenhuis bracket*.

It determines a vector bundle morphism $\Lambda^\sharp : T^*P \rightarrow TP$, defined by

$$\Lambda(\eta, \zeta) = \langle \zeta, \Lambda^\sharp(\eta) \rangle,$$

where η and $\zeta \in T^*P$ are two covectors attached to the same point in P .

4. The Hamiltonian formalism

4.6. The Hamiltonian formalism on Poisson manifolds (3)

Let (P, Λ) be a Poisson manifold. A (maybe time-dependent) vector field on P can be associated to each (maybe time-dependent) smooth function $H : \mathbb{R} \times P \rightarrow \mathbb{R}$. It is called the *Hamiltonian vector field* associated to the *Hamiltonian* H , and denoted by X_H . Its expression is

$$X_H(t, x) = \Lambda^\sharp(x)(dH_t(x)),$$

where $dH_t(x) = dH(t, x) - \frac{\partial H(t, x)}{\partial t} dt$ is the differential of the function deduced from H by considering t as a parameter with respect to which no differentiation is made.

The *Hamilton equation* determined by the (maybe time-dependent) Hamiltonian H is

$$\frac{d\varphi(t)}{dt} = X_H((t, \varphi(t))) = \Lambda^\sharp(dH_t)(\varphi(t)).$$

5. Hamiltonian symmetries

5.1. Presymplectic, symplectic and Poisson diffeomorphisms

Let M be a manifold endowed with some structure, which can be either

- a *presymplectic structure*, determined by a presymplectic form, i.e., a 2-form ω which is closed ($d\omega = 0$),
- a *symplectic structure*, determined by a symplectic form ω , i.e., a 2-form ω which is both closed ($d\omega = 0$) and nondegenerate ($\ker \omega = \{0\}$),
- a *Poisson structure*, determined by a smooth Poisson bivector field Λ satisfying $[\Lambda, \Lambda] = 0$.

Definition

A *presymplectic* (resp. *symplectic*, resp. *Poisson*) diffeomorphism of a presymplectic (resp., symplectic, resp. Poisson) manifold (M, ω) (resp. (M, Λ)) is a smooth diffeomorphism $f : M \rightarrow M$ such that $f^*\omega = \omega$ (resp. $f^*\Lambda = \Lambda$).

5. Hamiltonian symmetries

5.2. Presymplectic, symplectic and Poisson vector fields

Definition

A smooth vector field X on a presymplectic (resp. symplectic, resp. Poisson) manifold (M, ω) (resp. (M, Λ)) is said to be a *presymplectic* (resp. *symplectic*, resp. *Poisson*) vector field if $\mathcal{L}(X)\omega = 0$ (resp. if $\mathcal{L}(X)\Lambda = 0$), where $\mathcal{L}(X)$ denotes the Lie derivative of forms or multivector fields with respect to X .

Definition

Let (M, ω) be a presymplectic or symplectic manifold. A smooth vector field X on M is said to be *Hamiltonian* if there exists a smooth function $H : M \rightarrow \mathbb{R}$, called a *Hamiltonian* for X , such that

$$i(X)\omega = -dH.$$

Not any smooth function on a presymplectic manifold can be a Hamiltonian.

5. Hamiltonian symmetries

5.2. Presymplectic, symplectic and Poisson vector fields (2)

Definition

Let (M, Λ) be a Poisson manifold. A smooth vector field X on M is said to be *Hamiltonian* if there exists a smooth function $H \in C^\infty(M, \mathbb{R})$, called a *Hamiltonian* for X , such that $X = \Lambda^\sharp(dH)$. An equivalent definition is that

$$i(X)dg = \{H, g\} \quad \text{for any } g \in C^\infty(M, \mathbb{R}),$$

where $\{H, g\} = \Lambda(dH, dg)$ denotes the Poisson bracket of the functions H and g .

On a symplectic or a Poisson manifold, any smooth function can be a Hamiltonian.

Proposition

A Hamiltonian vector field on a presymplectic (resp. symplectic, resp. Poisson) manifold automatically is a presymplectic (resp. symplectic, resp. Poisson) vector field.

5. Hamiltonian symmetries

5.3. Lie algebras and Lie groups actions

An *action on the left* (resp. an *action on the right*) of a Lie group G on a smooth manifold M is a smooth map $\Phi : G \times M \rightarrow M$ (resp. a smooth map $\Psi : M \times G \rightarrow M$) such that

- for each fixed $g \in G$, the map $\Phi_g : M \rightarrow M$ defined by $\Phi_g(x) = \Phi(g, x)$ (resp. the map $\Psi_g : M \rightarrow M$ defined by $\Psi_g(x) = \Psi(x, g)$) is a smooth diffeomorphism of M ,
- $\Phi_e = \text{id}_M$ (resp. $\Psi_e = \text{id}_M$), e being the neutral element of G ,
- for each pair $(g_1, g_2) \in G \times G$, $\Phi_{g_1} \circ \Phi_{g_2} = \Phi_{g_1 g_2}$ (resp. $\Psi_{g_1} \circ \Psi_{g_2} = \Psi_{g_2 g_1}$).

An *action* of a Lie algebra \mathcal{G} on a smooth manifold M is a *Lie algebras morphism* of \mathcal{G} into the Lie algebra $A^1(M)$ of smooth vector fields on M , i.e. a map $\psi : \mathcal{G} \rightarrow A^1(M)$ which associates to each $X \in \mathcal{G}$ a smooth vector field $\psi(X)$ on M such that for each pair $(X, Y) \in \mathcal{G} \times \mathcal{G}$, $\psi([X, Y]) = [\psi(X), \psi(Y)]$.

5. Hamiltonian symmetries

5.3. Lie algebras and Lie groups actions (2)

An action Ψ , either on the left or on the right, of a Lie group G on a smooth manifold M automatically determines an action of its Lie algebra \mathcal{G} on that manifold, which associates to each $X \in \mathcal{G}$ the vector field $\psi(X)$ on M defined by

$$\psi(X)(x) = \frac{d}{ds} ((\Psi_{\exp(sX)})(x)) \Big|_{s=0}, \quad x \in M,$$

with the following convention: ψ a Lie algebras homomorphism when we take for Lie algebra \mathcal{G} of the Lie group G the Lie algebra or *right invariant* vector fields on G if Ψ is an action on the left, and the Lie algebra of *left invariant* vector fields on G if Ψ is an action on the right.

When M is a presymplectic (resp. symplectic, resp. Poisson) manifold, an action Ψ of a Lie group on M is called a presymplectic (resp. symplectic, resp. Poisson) action if for each $g \in G$, Ψ_g is a presymplectic (resp. symplectic, resp. Poisson) diffeomorphism of M . Similar definitions hold for Lie algebras actions.

5. Hamiltonian symmetries

5.4. Hamiltonian actions

Definitions

An action ψ of a Lie algebra \mathcal{G} on a presymplectic or symplectic manifold (M, ω) , or on a Poisson manifold (M, Λ) , is said to be *Hamiltonian* if for each $X \in \mathcal{G}$, the vector field $\psi(X)$ on M is Hamiltonian.

An action Ψ (either on the left or on the right) of a Lie group G on a presymplectic or symplectic manifold (M, ω) , or on a Poisson manifold (M, Λ) , is said to be *Hamiltonian* if that action is presymplectic, or symplectic, or Poisson (according to the structure of M), and if in addition the associated action of the Lie algebra \mathcal{G} of G is Hamiltonian.

5. Hamiltonian symmetries

5.5. Momentum maps of a Hamiltonian action

Proposition

Let ψ be a Hamiltonian action of a finite-dimensional Lie algebra \mathcal{G} on a presymplectic, symplectic or Poisson manifold (M, ω) or (M, Λ) . There exists a smooth map $J : M \rightarrow \mathcal{G}^*$, taking its values in the dual space \mathcal{G}^* of the Lie algebra \mathcal{G} , such that for each $X \in \mathcal{G}$ the Hamiltonian vector field $\psi(X)$ on M admits as Hamiltonian the function $J_X : M \rightarrow \mathbb{R}$, defined by

$$J_X(x) = \langle J(x), X \rangle, \quad x \in M.$$

The map J is called a **momentum map** for the Lie algebra action ψ . When ψ is the action of the Lie algebra \mathcal{G} of a Lie group G associated to a Hamiltonian action Ψ of a Lie group G , J is called a **momentum map** for the Hamiltonian Lie group action Ψ .

5. Hamiltonian symmetries

5.5. Momentum maps of a Hamiltonian action (2)

The momentum map J is not unique:

- when (M, ω) is a connected presymplectic or symplectic manifold, J is determined up to addition of an arbitrary constant element in \mathcal{G}^* ;
- when (M, Λ) is a connected Poisson manifold, the momentum map J is determined up to addition of an arbitrary \mathcal{G}^* -valued smooth map which, coupled with any $X \in \mathcal{G}$, yields a Casimir of the Poisson algebra of (M, Λ) , *i.e.* a smooth function on M whose Poisson bracket with any other smooth function on that manifold is the function identically equal to 0.

5. Hamiltonian symmetries

5.6. Noether's theorem in Hamiltonian formalism

Theorem (Noether's theorem in Hamiltonian formalism)

Let X_H and Z be two Hamiltonian vector fields on a presymplectic or symplectic manifold (M, ω) , or on a Poisson manifold (M, Λ) , which admit as Hamiltonians the smooth functions H and g on the manifold M . The function H remains constant on each integral curve of Z if and only if g remains constant on each integral curve of X_H .

Corollary (of Noether's theorem in Hamiltonian formalism)

Let $\psi : \mathcal{G} \rightarrow A^1(M)$ be a Hamiltonian action of a finite-dimensional Lie algebra \mathcal{G} on a presymplectic or symplectic manifold (M, ω) , or on a Poisson manifold (M, Λ) , and let $J : M \rightarrow \mathcal{G}^$ be a momentum map of this action. Let X_H be a Hamiltonian vector field on M admitting as Hamiltonian a smooth function H . If for each $X \in \mathcal{G}$ we have $i(\psi(X))(dH) = 0$, the momentum map J remains constant on each integral curve of X_H .*

5. Hamiltonian symmetries

5.7. Symplectic cocycles

Theorem (J.M. Souriau)

Let Φ be a Hamiltonian action (either on the left or on the right) of a Lie group G on a symplectic manifold (M, ω) and $J : M \rightarrow \mathcal{G}^$ be a moment map of this action. There exists an affine action A (either on the left or on the right) of the Lie group G on the dual \mathcal{G}^* of its Lie algebra \mathcal{G} such that the momentum map J is equivariant with respect to the actions of G Φ on M and A on \mathcal{G}^* :*

$$J \circ \Phi_g(x) = A_g \circ J(x) \quad \text{for all } g \in G, x \in M.$$

The action A can be written, with $g \in G$ and $\xi \in \mathcal{G}^$,*

$$\begin{cases} A(g, \xi) = \text{Ad}_{g^{-1}}^*(\xi) + \theta(g) & \text{if } \Phi \text{ is an action on the left,} \\ A(\xi, g) = \text{Ad}_g^*(\xi) - \theta(g^{-1}) & \text{if } \Phi \text{ is an action on the right.} \end{cases}$$

5. Hamiltonian symmetries

5.7. Symplectic cocycles (2)

Proposition

Under the assumptions and with the notations of the previous theorem, the map $\theta : G \rightarrow \mathcal{G}^$ is a cocycle of the Lie group G with values in \mathcal{G}^* , for the coadjoint representation. It means that it satisfies, for all g and $h \in G$,*

$$\theta(gh) = \theta(g) + \text{Ad}_g^*(\theta(h)).$$

More precisely θ is a symplectic cocycle. It means that its differential $T_e\theta : T_eG \equiv \mathcal{G} \rightarrow \mathcal{G}^$ at the neutral element $e \in G$ can be considered as a skew-symmetric bilinear form on \mathcal{G} :*

$$\Theta(X, Y) = \langle T_e\theta(X), Y \rangle = -\langle T_e\theta(Y), X \rangle.$$

5. Hamiltonian symmetries

5.7. Symplectic cocycles (3)

The bilinear form Θ on the Lie algebra \mathcal{G} is a symplectic cocycle of that Lie algebra. It means that it is skew-symmetric and satisfies, for all X, Y and $Z \in \mathcal{G}$,

$$\Theta([X, Y], Z) + \Theta([Y, Z], X) + \Theta([Z, X], Y) = 0.$$

Proposition

The composition law which associates to each pair (f, g) of smooth real-valued functions on \mathcal{G}^ the function $\{f, g\}_\Theta$ given by*

$$\{f, g\}_\Theta(x) = \langle x, [df(x), dg(x)] \rangle - \Theta(df(x), dg(x)), \quad x \in \mathcal{G}^*,$$

*(\mathcal{G} being identified with its bidual \mathcal{G}^{**}), determines a Poisson structure on \mathcal{G}^* , and the momentum map $J : M \rightarrow \mathcal{G}^*$ is a Poisson map, M being endowed with the Poisson structure associated to its symplectic structure.*

5. Hamiltonian symmetries

5.7. Symplectic cocycles (4)

When the momentum map J is replaced by another momentum map $J' = J + \mu$, where $\mu \in \mathcal{G}^*$ is a constant, the symplectic Lie group cocycle θ and the symplectic Lie algebra cocycle Θ are replaced by θ' and Θ' , respectively, given by

$$\begin{aligned}\theta'(g) &= \theta(g) + \mu - \text{Ad}_{g^{-1}}^*(\mu), \quad g \in G, \\ \Theta'(X, Y) &= \Theta(X, Y) + \langle \mu, [X, Y] \rangle, \quad X \text{ and } Y \in \mathcal{G}.\end{aligned}$$

These formulae show that $\theta' - \theta$ and $\Theta' - \Theta$ are *symplectic coboundaries* of the Lie group G and the Lie algebra \mathcal{G} . In other words, the *cohomology classes* of the cocycles θ and Θ only depend on the Hamiltonian action Φ of G on the symplectic manifold (M, ω) .

5. Hamiltonian symmetries

5.8. First application: symmetries of the phase space

Hamiltonian Symmetries are often used for the search of solutions of the equations of motion of mechanical systems. The symmetries considered are those of the *phase space* of the mechanical system. This space is very often a *symplectic manifold*, either the cotangent bundle to the configuration space with its canonical symplectic structure, or a more general symplectic manifold. Sometimes, after some simplifications, the phase space is a *Poisson manifold*.

The *Marsden-Weinstein reduction procedure* [25, 26] or one of its generalizations [28] is the most often method used to facilitate the determination of solutions of the equations of motion. In a first step, a possible value of the momentum map is chosen and the subset of the phase space on which the momentum map takes this value is determined. In a second step, that subset (when it is a smooth manifold) is quotiented by its isotropic foliation. The quotient manifold is a symplectic manifold of a dimension smaller than that of the original phase space, and one has an easier to solve Hamiltonian system on that reduced phase space.

5. Hamiltonian symmetries

5.8. First application: symmetries of the phase space (2)

When Hamiltonian symmetries are used for the reduction of the dimension of the phase space of a mechanical system, the symplectic cocycle of the Lie group of symmetries action, or of the Lie algebra of symmetries action, is almost always the *zero cocycle*.

For example, if the group of symmetries is the canonical lift to the cotangent bundle of a group of symmetries of the configuration space, not only the *canonical symplectic form*, but the *Liouville 1-form* of the cotangent bundle itself remains invariant under the action of the symmetry group, and this fact implies that the symplectic cohomology class of the action is zero.

A completely different way of using symmetries was initiated by Jean-Marie Souriau, who proposed to consider the symmetries of the *manifold of motions* of the mechanical system.

5. Hamiltonian symmetries

5.9. Second application: symmetries of the space of motions

Jean-Marie Souriau observed that the Lagrangian and Hamiltonian formalisms, in their usual formulations, *involve the choice of a particular reference frame*, in which the motion is described. This choice *destroys a part of the natural symmetries of the system*.

For example, in classical (non-relativistic) Mechanics, the natural symmetry group of an isolated mechanical system must contain the symmetry group of the *Galilean space-time*, called the *Galilean group*. This group is of *dimension 10*. It contains not only the *group of Euclidean displacements of space* which is of *dimension 6* and the *group of time translations* which is of *dimension 1*, but the *group of linear changes of Galilean reference frames* which is of *dimension 3*.

If we use the *Lagrangian formalism* or the *Hamiltonian formalism*, the Lagrangian or the Hamiltonian of the system *depends on the reference frame: it is not invariant with respect to linear changes of Galilean reference frames*.

5. Hamiltonian symmetries

5.9. Second application: symmetries of the space of motions (2)

It may seem strange to consider the set of all possible motions of a system, **which is unknown** as long as we have not determined all these possible motions. One may ask if it is really useful when we want to determine not all possible motions, but only one motion with prescribed initial data, since that motion is just one point of the (unknown) manifold of motion!

Souriau's answers to this objection are the following.

1. We know that the manifold of motions has a *symplectic structure*, and very often many things are known about its *symmetry properties*.
2. In classical (non-relativistic) mechanics, there exists a natural mathematical object which *does not depend on the choice of a particular reference frame* (even if the descriptions given to that object by different observers depend on the reference frame used by these observers): it is the *evolution space* of the system.

5. Hamiltonian symmetries

5.9. Second application: symmetries of the space of motions (3)

The knowledge of the equations which govern the system's evolution allows the full mathematical description of the *evolution space*, even when these equations are not yet solved.

Moreover, the symmetry properties of the *evolution space* are the same as those of the manifold of motions.

For example, the *evolution space* of a classical mechanical system with configuration manifold N is

- 1 in the *Lagrangian formalism*, the space $\mathbb{R} \times TN$ endowed with the presymplectic form $d\widehat{\omega}_L$, whose kernel is of dimension 1 when the Lagrangian L is hyper-regular,
- 2 in the Hamiltonian formalism, the space $\mathbb{R} \times T^*N$ with the presymplectic form $d\widehat{\omega}_H$, whose kernel is also of dimension 1.

The Poincaré-Cartan 1-form $\widehat{\omega}_L$ in the Lagrangian formalism or $\widehat{\omega}_H$ in the Hamiltonian formalism depend on the choice of a particular reference frame, made for using the Lagrangian or the Hamiltonian formalism.

5. Hamiltonian symmetries

5.9. Second application: symmetries of the space of motions (4)

But their exterior differentials, the presymplectic forms $d\widehat{\omega}_L$ or $d\widehat{\omega}_H$, *do not depend on that choice*, modulo a simple change of variables in the evolution space.

Souriau defined this presymplectic form in a framework more general than those of Lagrangian or Hamiltonian formalisms, and called it the *Lagrange form*. In this more general setting, it may not be an exact 2-form. Souriau proposed as a new *Principle*, the assumption that it always projects on the space of motions of the systems as a *symplectic form*, even in Relativistic Mechanics in which the definition of an evolution space is not clear. He called this new principle the *Maxwell Principle*.

V. Bargmann proved that the symplectic cohomology of the Galilean group is of dimension 1, and Souriau proved that the cohomology class of its action on the manifold of motions of an isolated classical (non-relativistic) mechanical system can be identified with the *total mass* of the system.

6. Souriau thermodynamics on Lie groups

6.1. Statistical states

Let N be the configuration manifold of a Lagrangian system whose Lagrangian $L : TN \rightarrow \mathbb{R}$ is hyper-regular and does not explicitly depend on the time t . Let $H : T^*N \rightarrow \mathbb{R}$ be the corresponding Hamiltonian and (M, ω) be the *manifold of motions* of the system. In the Hamiltonian formalism, a motion $\varphi \in M$ is a smooth curve $t \mapsto \varphi(t)$ defined on an open interval of \mathbb{R} , with values in T^*N . For each $t \in \mathbb{R}$, the map $\varphi \mapsto \varphi(t)$ is a symplectomorphism of the open subset of (M, ω) made by all motions defined on an interval containing t , onto an open subset of the phase space $(T^*N, d\theta_N)$. For simplicity I will assume in the following that this symplectomorphism is a *global symplectomorphism* of (M, ω) onto $(T^*N, d\theta_N)$. In other words I assume that *all the motions of the system are defined for all values of the time $t \in \mathbb{R}$* .

Definition

A *statistical state* of the mechanical system is a probability measure on the symplectic manifold of motions (M, ω) .

6. Souriau thermodynamics on Lie groups

6.1. Statistical states (2)

For simplicity I only consider in what follows statistical states which can be represented by a *smooth density of probability* $\rho : M \rightarrow [0, +\infty[$ with respect to *natural volume form* ω^n of the symplectic manifold of motions (M, ω) (with $n = \dim N$). We must therefore have

$$\int_M \rho(\varphi) \omega^n(\varphi) = 1.$$

With each statistical state with a smooth probability density ρ let us associate the number

$$s(\rho) = - \int_M \log(\rho(\varphi)) \rho(\varphi) \omega^n(\varphi),$$

with the convention that if $x \in M$ is such that $\varphi(x) = 0$, $\log(\varphi(x))\varphi(x) = 0$.

The Hamiltonian $H : T^*N \rightarrow \mathbb{R}$ remains constant along each motion of the system.

6. Souriau thermodynamics on Lie groups

6.2. Action of the group of time translations

Therefore we can define on the symplectic manifold of motions (M, ω) a smooth function $E : M \rightarrow \mathbb{R}$, called the *energy function*

$$E(\varphi) = H(\varphi(t)) \quad \text{for all } t \in \mathbb{R}, \quad \varphi \in M.$$

The Hamiltonian vector field X_E on M is the infinitesimal generator of the 1-dimensional *group of time translations*. A time translation $\Delta t : \mathbb{R} \rightarrow \mathbb{R}$ is a map $\Delta t : \mathbb{R} \rightarrow \mathbb{R}$, $\Delta t(t) = t + \Delta t$. The group of time translations can be identified with \mathbb{R} . It *acts on the manifold of motions* M by the action Φ^E , such that for each time translation Δt and each motion φ , $\Phi_{\Delta t}^E(\varphi)$ is the motion

$$t \mapsto \Phi_{\Delta t}^E(\varphi)(t) = \varphi(t + \Delta t).$$

Following the ideas of *Ludwig Boltzmann* (1844–1906), more recently reformulated by *E.T. Jaynes* [12] and *G.W. Mackey* [21], *J.-M. Souriau* [30] proposed the following definition of a *thermodynamic equilibrium state*.

6. Souriau thermodynamics on Lie groups

6.3. Thermodynamic equilibrium state

Definition

A *thermodynamic equilibrium state* of the mechanical system, for a given value mean value Q of the energy function E , is a statistical state with a smooth probability density $\rho \geq 0$ satisfying the two constraints

$$\int_M \rho(\varphi) \omega^n(\varphi) = 1,$$
$$\int_M \rho(\varphi) E(\varphi) \omega^n(\varphi) = Q,$$

which, moreover, is such that the integral

$$s(\rho) = - \int_M \log(\rho(\varphi)) \rho(\varphi) \omega^n(\varphi)$$

is stationary with respect to all infinitesimal smooth variations of the probability density $\rho \geq 0$ submitted to these two constraints.

6. Souriau thermodynamics on Lie groups

6.3. Thermodynamic equilibrium state (2)

By using the standard techniques of calculus of variations, Souriau proves that for each mean value Q of the energy function for which the involved integrals are normally convergent, there exists a unique thermodynamic equilibrium state whose probability density ρ is given by

$$\rho(\varphi) = \exp(-\Psi - \Theta \cdot E(\varphi)) ,$$

where Ψ and Θ are two constants which satisfy the two equalities

$$\begin{aligned} \Psi &= \log \left(\int_M \exp(-\Theta \cdot E(\varphi)) \omega^n(\varphi) \right) , \\ Q &= \frac{\int_M E(\varphi) \exp(-\Theta \cdot E(\varphi)) \omega^n(\varphi)}{\int_M \exp(-\Theta \cdot E(\varphi)) \omega^n(\varphi)} . \end{aligned}$$

6. Souriau thermodynamics on Lie groups

6.3. Thermodynamic equilibrium state (3) Souriau proves that these two equalities imply that Ψ and Q are smooth functions of the variable Θ , and that

$$Q(\Theta) = -\frac{d\Psi(\Theta)}{d\Theta}.$$

Moreover, by using convexity arguments, he proves that when Q is given, there is at most one corresponding value of Θ , so that $\Psi(\Theta)$ and the probability density ρ are uniquely determined. Moreover, he proves that the value of $s(\rho)$ is a strict maximum, with respect to smooth variations of ρ satisfying the two above stated constraints. That maximum is a function S of the variable Θ given by

$$S(\Theta) = \Psi(\Theta) + \Theta \cdot Q(\Theta), \quad \text{therefore}$$

$$\frac{dS(\Theta)}{d\Theta} = -\Theta \frac{d^2\Psi(\Theta)}{d\Theta^2}.$$

6. Souriau thermodynamics on Lie groups

6.3. Thermodynamic equilibrium state (4)

Souriau proves that $\frac{d^2\Psi(\Theta)}{d\Theta^2} > 0$, therefore Ψ is a convex function.

Physical interpretation of these results: Θ is related to the *absolute temperature* T by

$$\Theta = \frac{1}{kT},$$

where k is the *Boltzmann constant*, S is the *entropy* and Q the *internal energy* of the system. By this means Souriau recovers the *Maxwell distribution of velocities* of particles in a perfect gas.

6. Souriau thermodynamics on Lie groups

6.4. Generalization for a Hamiltonian Lie group action

The energy function E on the symplectic manifold of motions (M, ω) can be seen as the *momentum map* of the Hamiltonian action Φ^E on that manifold of the *one-dimensional Lie group of time translations*.

Souriau proposes a natural generalization of the definition of a thermodynamic equilibrium state in which a (maybe *multi-dimensional* and maybe *non-Abelian*) Lie group G acts, by a Hamiltonian action Φ , on that symplectic manifold. Let \mathcal{G} be the Lie algebra of G , \mathcal{G}^* be its dual space and $J : M \rightarrow \mathcal{G}^*$ be a momentum map of the action Φ . In [30], he calls it an *equilibrium state allowed by the group G* and in his later papers and book [31, 32] a *Gibbs state of the Lie group G* , probably because it is not so clear whether physically such a state really is an equilibrium.

6. Souriau thermodynamics on Lie groups

6.4. Generalization for a Hamiltonian Lie group action (2)

Definition

A *Gibbs state* of a connected Lie group G acting on a connected symplectic manifold (M, ω) by a Hamiltonian action Φ , with a momentum map $J : M \rightarrow \mathcal{G}^*$, for a given value mean value $Q \in \mathcal{G}^*$ of that momentum map, is a statistical state with a smooth probability density $\rho \geq 0$ satisfying the two constraints

$$\int_M \rho(\varphi) \omega^n(\varphi) = 1, \quad \int_M \rho(\varphi) J(\varphi) \omega^n(\varphi) = Q,$$

which, moreover, is such that the integral

$$s(\rho) = - \int_M \log(\rho(\varphi)) \rho(\varphi) \omega^n(\varphi)$$

is stationary with respect to all infinitesimal smooth variations of the probability density $\rho \geq 0$ submitted to these two constraints.

6. Souriau thermodynamics on Lie groups

6.4. Generalization for a Hamiltonian Lie group action (3)

By the same calculations as those made for a thermodynamic equilibrium, Souriau obtains the following results. For each value Q of the momentum map J for which the involved integrals are normally convergent, there exists a unique Gibbs state whose probability density ρ is given by

$$\rho(\varphi) = \exp\left(-\Psi - \langle \Theta, J(\varphi) \rangle\right),$$

where Ψ is a real constant and Θ a constant which takes its value in the Lie algebra \mathcal{G} , considered as the dual of \mathcal{G}^* , which satisfy the two equalities

$$\begin{aligned}\Psi &= \log\left(\int_M \exp\left(-\langle \Theta, J(\varphi) \rangle\right) \omega^n(\varphi)\right), \\ Q &= \frac{\int_M J(\varphi) \exp\left(-\langle \Theta, E(\varphi) \rangle\right) \omega^n(\varphi)}{\int_M \exp\left(-\langle \Theta, J(\varphi) \rangle\right) \omega^n(\varphi)}.\end{aligned}$$

6. Souriau thermodynamics on Lie groups

6.4. Generalization for a Hamiltonian Lie group action (4) Souriau proves that these two equalities imply that Ψ and Q are smooth functions of the variable $\Theta \in \mathcal{G}$, which take their value, respectively, in \mathbb{R} and in \mathcal{G}^* , and that

$$Q(\Theta) = -D\Psi(\Theta),$$

where $D\Psi$ is the *first differential* of $\Psi : \mathcal{G} \rightarrow \mathbb{R}$. Exactly as for an equilibrium state, when Q is given, there is at most one corresponding value of Θ , so that $\Psi(\Theta)$ and the probability density ρ are uniquely determined. Moreover, he proves that the value of $s(\rho)$ is a strict maximum, with respect to smooth variations of ρ satisfying the two above stated constraints. That maximum is a function S of the variable Θ given by

$$S(\Theta) = \Psi(\Theta) + \langle \Theta, Q(\Theta) \rangle.$$

The *second differential* $D^2\Psi$ of the function $\Psi : \mathcal{G} \rightarrow \mathbb{R}$ is a *positive symmetric bilinear form*, which moreover is *definite* except when J takes its value in an affine subspace of \mathcal{G}^* .

6. Souriau thermodynamics on Lie groups

6.4. Generalization for a Hamiltonian Lie group action (5)

When (M, ω) is the manifold of motions of a mechanical system, Θ is interpreted as a *\mathcal{G} -valued generalized temperature* and $S(\rho)$ as the *entropy function* of the Gibbs state ρ .

There is however an *important difference* between a thermodynamic equilibrium state and a Gibbs state of a Lie group G : a Gibbs state *may not be invariant* with respect to the action of the Lie group G on the symplectic manifold of motions (M, ω) , since the expression of its probability density ρ involves the value of the momentum map J , which is *equivariant* with respect to the action Φ of G on (M, ω) and an *affine action* of G on the dual of its Lie algebra \mathcal{G}^* , whose linear part is the coadjoint action, eventually with a *symplectic cocycle* of G .

Moreover, there are Hamiltonian actions for which *the set of Gibbs states is empty* because the involved integrals never converge: this happens, for example, for the action of the *Galilean group* on the manifold of motions of an isolated classical mechanical system.

6. Souriau thermodynamics on Lie groups

6.4. Generalization for a Hamiltonian Lie group action (6)

In [32], Souriau presents several examples, both for classical and for relativistic systems, which have clear physical interpretations. For example he discusses both *non-relativistic and relativistic centrifuges* for isotopic separation, and recovers the *velocity distribution of particles in a relativistic perfect gas* which can be found in the book by J.L. Synge [34].

In the second part of that paper, he presents a very nice cosmological model of the Universe, founded on his ideas of thermodynamics of Lie groups, compatible with the observed *isotropy of the 2.7 Kelvin degrees microwave background radiation*.

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Appendix. The Euler-Poincaré equation

In a short Note [29] published in 1901, the great french mathematician *Henri Poincaré* (1854–1912) proposed a new formulation of the equations of Mechanics.

Assumptions

Let N be the configuration manifold of a conservative Lagrangian system, with a smooth Lagrangian $L : TN \rightarrow \mathbb{R}$ which does not depend explicitly on time. Poincaré assumes that there exists an homomorphism ψ of a finite-dimensional real Lie algebra \mathcal{G} into the Lie algebra $A^1(N)$ of smooth vector fields on N , such that for each $x \in N$, the values at x of the vector fields $\psi(X)$, when X varies in \mathcal{G} , completely fill the tangent space $T_x N$. The action ψ is then said to be *locally transitive*.

Appendix. The Euler-Poincaré equation (2)

Of course these assumptions imply $\dim \mathcal{G} \geq \dim N$

Under these assumptions, [Henri Poincaré](#) proved that the equations of motion of the Lagrangian system could be written on $N \times \mathcal{G}$ or on $N \times \mathcal{G}^*$, where \mathcal{G}^* is the dual of the Lie algebra \mathcal{G} , instead of on the tangent bundle TN . When $\dim \mathcal{G} = \dim N$ (which can occur only when the tangent bundle TN is trivial) the obtained equation, called the *Euler-Poincaré equation*, is *perfectly equivalent to the Euler-Lagrange equations* and may, in certain cases, be easier to use.

But when $\dim \mathcal{G} > \dim N$, the system made by the Euler-Poincaré equation is *underdetermined*.

Appendix. The Euler-Poincaré equation (3)

Let $\gamma : [t_0, t_1] \rightarrow N$ be a smooth parametrized curve in N . Poincaré proves that there exists a smooth curve $V : [t_0, t_1] \rightarrow \mathcal{G}$ in the Lie algebra \mathcal{G} such that, for each $t \in [t_0, t_1]$,

$$\psi(V(t))(\gamma(t)) = \frac{d\gamma(t)}{dt}. \quad (*)$$

When $\dim \mathcal{G} > \dim N$ the smooth curve V in \mathcal{G} is not uniquely determined by the smooth curve γ in N . However, instead of writing the second-order Euler-Lagrange differential equations on TN satisfied by γ when this curve is a possible motion of the Lagrangian system, Poincaré derives a *first order differential equation for the curve V* and proves that it is satisfied, together with Equation (*), *if and only if γ is a possible motion of the Lagrangian system.*

Let $\varphi : N \times \mathcal{G} \rightarrow TN$ and $\bar{L} : N \times \mathcal{G} \rightarrow \mathbb{R}$ be the maps

$$\varphi(x, X) = \psi(X)(x), \quad \bar{L}(x, X) = L \circ \varphi(x, X).$$

Appendix. The Euler-Poincaré equation (4)

We denote by $d_1\bar{L} : N \times \mathcal{G} \rightarrow T^*N$ and by $d_2\bar{L} : N \times \mathcal{G} \rightarrow \mathcal{G}^*$ the partial differentials of $\bar{L} : N \times \mathcal{G} \rightarrow \mathbb{R}$ with respect to its first variable $x \in N$ and with respect to its second variable $X \in \mathcal{G}$.

The map $\varphi : N \times \mathcal{G} \rightarrow TN$ is a *surjective vector bundles morphism* of the trivial vector bundle $N \times \mathcal{G}$ into the tangent bundle TN . Its *transpose* $\varphi^T : T^*N \rightarrow N \times \mathcal{G}^*$ is therefore an *injective vector bundles morphism*, which can be written

$$\varphi^T(\xi) = (\pi_N(\xi), J(\xi)),$$

where $\pi_N : T^*N \rightarrow N$ is the canonical projection of the cotangent bundle and $J : T^*N \rightarrow \mathcal{G}^*$ is a smooth map whose restriction to each fibre T_x^*N of the cotangent bundle is linear, and is the transpose of the map $X \mapsto \varphi(x, X) = \psi(X)(x)$. It can be seen that J is in fact a *Hamiltonian momentum map*.

Appendix. The Euler-Poincaré equation (5)

Let $\mathcal{L}_L = d_{\text{vert}}L : TN \rightarrow T^*N$ be the *Legendre map*.

Theorem (Euler-Poincaré equation)

With the above defined notations, let $\gamma : [t_0, t_1] \rightarrow N$ be a smooth parametrized curve in N and $V : [t_0, t_1] \rightarrow \mathcal{G}$ be a smooth parametrized curve such that, for each $t \in [t_0, t_1]$,

$$\psi(V(t))(\gamma(t)) = \frac{d\gamma(t)}{dt}. \quad (*)$$

The curve γ is a possible motion of the Lagrangian system if and only if V satisfies the equation

$$\left(\frac{d}{dt} - \text{ad}_{V(t)}^* \right) \left(J \circ \mathcal{L}_L \circ \varphi(\gamma(t), V(t)) \right) - J \circ d_1 \bar{L}(\gamma(t), V(t)) = 0. \quad (**)$$

Appendix. The Euler-Poincaré equation (6)

Remark

Equation (*) is called the *compatibility condition* and Equation (***) is the *Euler-Poincaré equation*. It can be written also as

$$\left(\frac{d}{dt} - \text{ad}_{V(t)}^* \right) \left(d_2 \bar{L}(\gamma(t), V(t)) \right) - J \circ d_1 \bar{L}(\gamma(t), V(t)) = 0. \quad (***)$$

Several examples of applications of the Euler-Poincaré equation can be found in [23, 24].

Appendix. The Euler-Poincaré equation (7)

When the function $\bar{L} : N \times \mathcal{G} \rightarrow \mathbb{R}$ *does not depend on its first variable* $x \in N$, we have $d_1 \bar{L} = 0$, and the Euler-Poincaré equation becomes simpler: it can be written either as

$$\left(\frac{d}{dt} - \text{ad}_{V(t)}^* \right) \left(J \circ \mathcal{L}_L \circ \varphi(\gamma(t), V(t)) \right) = 0,$$

or as

$$\left(\frac{d}{dt} - \text{ad}_{V(t)}^* \right) \left(d_2 \bar{L}(\gamma(t), V(t)) \right) = 0.$$

The condition that $\bar{L} : N \times \mathcal{G} \rightarrow \mathbb{R}$ does not depend on its first variable $x \in N$ *does not mean that the Lagrangian $L : TN \rightarrow \mathbb{R}$ is invariant by the canonical lift to TN of the action on N of the Lie algebra \mathcal{G}* . When the Lagrangian L is *hyper-regular*, it *does not mean that the Hamiltonian H_L associated to L is invariant by the canonical lift to T^*N of that action*. On the contrary, when in addition $\dim \mathcal{G} = \dim N$, *it means that the Hamiltonian H_L can be written as $H_L = H_{\mathcal{G}^*} \circ J$, where $H_{\mathcal{G}^*}$ is a smooth function defined on \mathcal{G}^** , and the Euler-Poincaré equation can be identified with a

*Hamilton equation on \mathcal{G}^**

References

Several books [1, 2, 5, 7, 10, 11, 19, 28, 31] present the mathematical tools used in Geometric Mechanics.

The calculus of variations and its applications in Mechanics are presented in [33, 2, 3, 4, 17, 22]. Poisson manifolds were defined by *A. Lichnerowicz* [20], considered in the more general setting of *local Lie algebras* by *A. Kirillov* [13]. Their local structure was studied by *A. Weinstein* [38]. Their geometric properties are extensively described in the more recent books [37, 18]. The best text about the Schouten-Nijenhuis bracket, in which the sign conventions used are the most natural and the easiest to use, is [15]. The *Bargmann group* and its applications in Thermodynamics are discussed in the recent paper by *G. de Saxcé and C. Vallée* [6]. The very nice recent book [14] by *Y. Kosmann-Schwarzbach* gives an excellent historical and mathematical presentation of the *Noether theorems*.

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