Dirac brackets and bihamiltonian structures

Charles-Michel Marle

cmm1934@orange.fr

Université Pierre et Marie Curie

Paris, France

Summary (1)

Introduction

Lagrangian formalism

Lagrangian formalism and Legendre map

Generalized Hamiltonian formalism

Primary constraints
Generalized Hamiltonian dynamics
Secondary constraints
Dirac's algorithm
Final constraint submanifold
First and second class constraints
Poisson brackets of second class constraints
Symplectic explanation

Summary (2)

- The Dirac bracket
  - The Poisson-Dirac bivector
  - Compatibility of Λ anΛ_D
- Example
- Thanks

Introduction

Around 1950, P. Dirac developed a *Generalized Hamiltonian dynamics* for Lagrangian systems with degenerate Lagrangians. In this theory, the phase space of the system (i.e. the cotangent bundle to the configuration manifold) is endowed with two Poisson brackets:

Introduction

Around 1950, P. Dirac developed a *Generalized Hamiltonian dynamics* for Lagrangian systems with degenerate Lagrangians. In this theory, the phase space of the system (i.e. the cotangent bundle to the configuration manifold) is endowed with two Poisson brackets:

- the usual Poisson bracket associated to its symplectic structure,

Introduction

Around 1950, P. Dirac developed a *Generalized Hamiltonian dynamics* for Lagrangian systems with degenerate Lagrangians. In this theory, the phase space of the system (i.e. the cotangent bundle to the configuration manifold) is endowed with two Poisson brackets:

- the usual Poisson bracket associated to its symplectic structure,
- a modified Poisson bracket (today known as the *Dirac bracket*), used by Dirac for the canonical quantization of the system.

Introduction

Around 1950, P. Dirac developed a *Generalized Hamiltonian dynamics* for Lagrangian systems with degenerate Lagrangians. In this theory, the phase space of the system (i.e. the cotangent bundle to the configuration manifold) is endowed with two Poisson brackets:

- the usual Poisson bracket associated to its symplectic structure,
- a modified Poisson bracket (today known as the *Dirac bracket*), used by Dirac for the canonical quantization of the system.

I what follows I will describe Dirac's theory of generalized Hamiltonian dynamics, and I will consider its links with the theory of bihamiltonian systems.

Lagrangian formalism

We consider a mechanical system with a smooth manifold Q as configuration space. The dynamical properties of the system are described by a smooth Lagrangian $L : TQ \to \mathbb{R}$. Possible motions of the system are curves $t \mapsto q(t)$, parametrized by the time t, defined on intervals $[t_0, t_1] \subset \mathbb{R}$, which are extremals of the action integral

$$I = \int_{t_0}^{t_1} L\left(\frac{dq(t)}{dt}\right)\,dt\,,$$

with fixed endpoints.

agrangian formalism and Legendre map (

The curve $t \mapsto q(t)$ is an extremal of the action integral if and only if it satisfies Lagrange equations, which, in a chart of Q with local coordinates $(q^1, \dots, q^n)$, and the associated chart of TQ with local coordinates $(q^1, \dots, q^n, \dot{q}^1, \dots, \dot{q}^n)$, are

$$\frac{d}{dt}\left(\frac{\partial L(q,\dot{q})}{\partial \dot{q}^i}\right) - \frac{\partial L(q,\dot{q})}{\partial q^i} = 0\,, \quad 1 \leq i \leq n\,.$$

agrangian formalism and Legendre map (

The curve $t \mapsto q(t)$ is an extremal of the action integral if and only if it satisfies Lagrange equations, which, in a chart of Q with local coordinates $(q^1, \dots, q^n)$, and the associated chart of TQ with local coordinates $(q^1, \dots, q^n, \dot{q}^1, \dots, \dot{q}^n)$, are

$$\frac{d}{dt}\left(\frac{\partial L(q,\dot{q})}{\partial \dot{q}^i}\right) - \frac{\partial L(q,\dot{q})}{\partial q^i} = 0\,, \quad 1 \leq i \leq n\,.$$

When the *Legendre map*

$$\mathcal{L} : TQ \to T^*Q\,, \quad (q,\dot{q}) \mapsto (q,p) \text{ with } p_i = \frac{\partial L(q,\dot{q})}{\partial \dot{q}^i}$$

is a (local) diffeomorphism, one may (locally) define a Hamiltonian H on T^*Q by setting

$$H(q,p) = \sum_{i=1}^{n} \dot{q}^i p_i - L(q,\dot{q})\,,$$

agrangian formalism and Legendre map (

where $\dot{q}^i$ and $(q, \dot{q}) = \mathcal{L}^{-1}(q,p)$ are expressed in terms of (q,p) by means of the (local) inverse $\mathcal{L}^{-1}$ of the Legendre map. Under these assumptions, Lagrange equations are (locally) equivalent to Hamilton equations,

$$\frac{dq^i}{dt} = \frac{\partial H(q,p)}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial H(q,p)}{\partial q^i}.$$

agrangian formalism and Legendre map (2

where $\dot{q}^i$ and $(q, \dot{q}) = \mathcal{L}^{-1}(q, p)$ are expressed in terms of (q, p) by means of the (local) inverse $\mathcal{L}^{-1}$ of the Legendre map. Under these assumptions, Lagrange equations are (locally) equivalent to Hamilton equations,

$$\frac{dq^i}{dt} = \frac{\partial H(q,p)}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial H(q,p)}{\partial q^i}.$$

When the Legendre map is not a (local) diffeomorphism, we still can define a Hamiltonian on $TQ \oplus T^*Q$ by

$$H(q, \dot{q}, p) = \sum_{i=1}^{n} \dot{q}^i p_i - L(q, \dot{q}).$$

That Hamiltonian *is not* a function defined on T^*Q.

Primary constraints (1)

Dirac *does not assume* that the Legendre map is a local diffeomorphism. Instead, he (tacitly) assumes that it is a map of constant rank $2n - r$, with $1 \leq r \leq n$. Let $D_0 = \mathcal{L}(TQ)$ be its image. Dirac assumes that there exist r smooth functions $\Phi_\alpha : T^*Q \to \mathbb{R}$, $1 \leq \alpha \leq r$, such that D_0 is defined by the equations

$$\Phi_\alpha = 0\,, \quad 1 \leq \alpha \leq r\,,$$

the $d\Phi_\alpha$ being linearly independent on D_0. These equations are called *primary constraints*. The functions Φ_α are called the *primary constraint functions*.

Primary constraints (2)

These assumptions are valid locally: since $\mathcal{L}$ is of constant rank $2n - r$, each point in TQ has an open neighbourhood U in TQ such that $\mathcal{L}(U)$ is a smooth $(2n - r)$-dimensional submanifold of T^*Q defined by equations of that form, the Φ_i being smooth functions defined on some open subset V of T^*Q containing $\mathcal{L}(U)$.

Primary constraints (2)

These assumptions are valid locally: since $\mathcal{L}$ is of constant rank $2n - r$, each point in TQ has an open neighbourhood U in TQ such that $\mathcal{L}(U)$ is a smooth $(2n - r)$-dimensional submanifold of T^*Q defined by equations of that form, the Φ_i being smooth functions defined on some open subset V of T^*Q containing $\mathcal{L}(U)$.

Globally, D_0 may not be a “true” submanifold of T^*Q: it may be self-intersecting, with multiple points.

Primary constraints (2)

These assumptions are valid locally: since $\mathcal{L}$ is of constant rank $2n-r$, each point in TQ has an open neighbourhood U in TQ such that $\mathcal{L}(U)$ is a smooth $(2n-r)$-dimensional submanifold of T^*Q defined by equations of that form, the Φ_i being smooth functions defined on some open subset V of T^*Q containing $\mathcal{L}(U)$.

Globally, D_0 may not be a “true” submanifold of T^*Q: it may be self-intersecting, with multiple points.

The partial derivatives of $H : TQ \oplus T^*Q \to \mathbb{R}$ are

$$\frac{\partial H(q,\dot q,p)}{\partial q^i} = -\frac{\partial L(q,\dot q)}{\partial q^i}\,, \qquad \frac{\partial H(q,\dot q,p)}{\partial \dot q^i} = p_i - \frac{\partial L(q,\dot q)}{\partial \dot q^i}\,,$$

$$\frac{\partial H(q,\dot q,p)}{\partial p_i} = \dot q^i\,.$$

Generalized Hamiltonian dynamics (1)

Let

$$\Gamma_{\mathcal{L}} = \left\{(q, \dot{q}, p) \in TQ \oplus T^*Q; (q, p) = \mathcal{L}(q, \dot{q})\right\}$$

be the graph of the Legendre map. We see that

$$\frac{\partial H(q, \dot{q}, p)}{\partial \dot{q}^i} = 0 \quad \text{when } (q, \dot{q}, p) \in \Gamma_{\mathcal{L}}\,.$$

Generalized Hamiltonian dynamics (1)

Let

$$\Gamma_{\mathcal{L}} = \big\{(q,\dot q,p) \in TQ \oplus T^*Q; (q,p) = \mathcal{L}(q,\dot q)\big\}$$

be the graph of the Legendre map. We see that

$$\frac{\partial H(q,\dot q,p)}{\partial \dot q^i} = 0 \quad \text{when } (q,\dot q,p) \in \Gamma_{\mathcal{L}}\,.$$

In other words, on the graph $\Gamma_{\mathcal{L}}$ (which is a $2n$-dimensional submanifold of the $3n$-dimensional manifold $TQ \oplus T^*Q$), the Hamiltonian H does not depend on the variables $\dot q^i$.

Generalized Hamiltonian dynamics (1)

Let

$$\Gamma_{\mathcal{L}} = \{(q, \dot q, p) \in TQ \oplus T^*Q; (q,p) = \mathcal{L}(q, \dot q)\}$$

be the graph of the Legendre map. We see that

$$\frac{\partial H(q, \dot q, p)}{\partial \dot q^i} = 0 \quad \text{when } (q, \dot q, p) \in \Gamma_{\mathcal{L}}\,.$$

In other words, on the graph $\Gamma_{\mathcal{L}}$ (which is a $2n$-dimensional submanifold of the $3n$-dimensional manifold $TQ \oplus T^*Q$), the Hamiltonian H does not depend on the variables $\dot q^i$.

Dirac considers that there exists a smooth function $\widehat{H}$, defined on an open subset of T^*Q containing D_0, such that

$$H(q, \dot q, p) = \widehat{H}(q, p) \quad \text{when } (q, \dot q, p) \in \Gamma_{\mathcal{L}}\,.$$

Generalized Hamiltonian dynamics (2)

The function $\widehat{H}$ with these properties, when it exists, is not unique: we may add to it any smooth function which vanishes on the image D_0 of the Legendre map.

Generalized Hamiltonian dynamics (2)

The function $\widehat{H}$ with these properties, when it exists, is not unique: we may add to it any smooth function which vanishes on the image D_0 of the Legendre map.

For the existence of $\widehat{H}$, we may have to restrict the Legendre map $\mathcal{L}$ to a suitable open subset of TQ, such that for each point $(q,p) \in D_0$, the intersection of that open subset with $\mathcal{L}^{-1}(q,p)$ is connected. Following Dirac we will assume that functions with these properties do exist, and we choose arbitrarily one of them.

Generalized Hamiltonian dynamics (2)

The function $\widehat{H}$ with these properties, when it exists, is not unique: we may add to it any smooth function which vanishes on the image D_0 of the Legendre map.

For the existence of $\widehat{H}$, we may have to restrict the Legendre map $\mathcal{L}$ to a suitable open subset of TQ, such that for each point $(q,p) \in D_0$, the intersection of that open subset with $\mathcal{L}^{-1}(q,p)$ is connected. Following Dirac we will assume that functions with these properties do exist, and we choose arbitrarily one of them.

Instead of $\widehat{H}$ and Φ_α, defined on T^*Q, it is more convenient to consider the functions, defined on $TQ \oplus T^*Q$,

$$\widetilde{H} = \widehat{H} \circ \pi_{T^*Q}\,, \quad \widetilde{\Phi}_\alpha = \Phi_\alpha \circ \pi_{T^*Q}\,, \quad 1 \leq \alpha \leq r\,,$$

$\pi_{T^*Q} : TQ \oplus T^*Q \to T^*Q$ being the canonical submersion.

Generalized Hamiltonian dynamics (3)

Both H and $\widetilde{H}$ are functions defined on $TQ \oplus T^*Q$, which are equal on the graph $\Gamma_{\mathcal{L}}$ of $\mathcal{L}$:

$$H(q, \dot{q}, p) = \widehat{H}(q, p) = \widetilde{H}(q, \dot{q}, p) \quad \text{when } (q, p) = \mathcal{L}(q, \dot{q}) \, .$$

Generalized Hamiltonian dynamics (3)

Both H and $\widetilde{H}$ are functions defined on $TQ \oplus T^*Q$, which are equal on the graph $\Gamma_{\mathcal{L}}$ of $\mathcal{L}$:

$$H(q, \dot{q}, p) = \widehat{H}(q, p) = \widetilde{H}(q, \dot{q}, p) \quad \text{when } (q, p) = \mathcal{L}(q, \dot{q})\,.$$

So for each $(q, \dot{q}, p) \in \Gamma_{\mathcal{L}}$, $dH(q, \dot{q}, p)$ and $d\widetilde{H}(q, \dot{q}, p)$ are equal on the subspace of vectors tangent to $\Gamma_{\mathcal{L}}$. Moreover,

$$\bigcap_{\alpha=1}^{r} \ker d\widetilde{\Phi}_\alpha(q, \dot{q}, p) \subset \ker d(H - \widetilde{H})(q, \dot{q}, p)\,.$$

Generalized Hamiltonian dynamics (3)

Both H and $\widetilde{H}$ are functions defined on $TQ \oplus T^*Q$, which are equal on the graph $\Gamma_{\mathcal{L}}$ of $\mathcal{L}$:

$$H(q,\dot{q},p) = \widehat{H}(q,p) = \widetilde{H}(q,\dot{q},p) \quad \text{when } (q,p) = \mathcal{L}(q,\dot{q})\,.$$

So for each $(q,\dot{q},p) \in \Gamma_{\mathcal{L}}$, $dH(q,\dot{q},p)$ and $d\widetilde{H}(q,\dot{q},p)$ are equal on the subspace of vectors tangent to $\Gamma_{\mathcal{L}}$. Moreover,

$$\bigcap_{\alpha=1}^{r} \ker d\widetilde{\Phi}_\alpha(q,\dot{q},p) \subset \ker d(H-\widetilde{H})(q,\dot{q},p)\,.$$

The theory of Lagrange multipliers shows that, for $(q,\dot{q},p) \in \Gamma_{\mathcal{L}}$,

$$dH(q,\dot{q},p) = d\widetilde{H}(q,\dot{q},p) + \sum_{\alpha=1}^{r} v^\alpha d\widetilde{\Phi}_\alpha(q,\dot{q},p)\,.$$

Generalized Hamiltonian dynamics (3)

Both H and $\widetilde{H}$ are functions defined on $TQ \oplus T^*Q$, which are equal on the graph $\Gamma_{\mathcal{L}}$ of $\mathcal{L}$:

$$H(q,\dot{q},p) = \widehat{H}(q,p) = \widetilde{H}(q,\dot{q},p) \quad \text{when } (q,p) = \mathcal{L}(q,\dot{q})\,.$$

So for each $(q,\dot{q},p) \in \Gamma_{\mathcal{L}}$, $dH(q,\dot{q},p)$ and $d\widetilde{H}(q,\dot{q},p)$ are equal on the subspace of vectors tangent to $\Gamma_{\mathcal{L}}$. Moreover,

$$\bigcap_{\alpha=1}^{r} \ker d\widetilde{\Phi}_\alpha(q,\dot{q},p) \subset \ker d(H - \widetilde{H})(q,\dot{q},p)\,.$$

The theory of Lagrange multipliers shows that, for $(q,\dot{q},p) \in \Gamma_{\mathcal{L}}$,

$$dH(q,\dot{q},p) = d\widetilde{H}(q,\dot{q},p) + \sum_{\alpha=1}^{r} v^\alpha d\widetilde{\Phi}_\alpha(q,\dot{q},p)\,.$$

Generalized Hamiltonian dynamics (4)

For each $(q, \dot{q}, p) \in \Gamma_{\mathcal{L}}$, the family of Lagrange multipliers $(v^{\alpha}\,,\ 1 \leq \alpha \leq r)$ depends on that point and of the time t, and may not be unique.

Generalized Hamiltonian dynamics (4)

For each $(q,\dot q,p)\in\Gamma_{\mathcal L}$, the family of Lagrange multipliers $(v^\alpha\,,\ 1\leq\alpha\leq r)$ depends on that point and of the time t, and may not be unique.

We have seen that, for $(q,\dot q,p)\in\Gamma_{\mathcal L}$,

$$\frac{\partial H(q,\dot q,p)}{\partial q^i}=-\frac{\partial L(q,\dot q)}{\partial q^i}\,,\qquad \frac{\partial H(q,\dot q,p)}{\partial \dot q^i}=0\,,\qquad \frac{\partial H(q,\dot q,p)}{\partial p_i}=\dot q^i\,.$$

Generalized Hamiltonian dynamics (4)

For each $(q, \dot q, p) \in \Gamma_{\mathcal{L}}$, the family of Lagrange multipliers $(v^\alpha\,,\ 1 \leq \alpha \leq r)$ depends on that point and of the time t, and may not be unique.

We have seen that, for $(q, \dot q, p) \in \Gamma_{\mathcal{L}}$,

$$\frac{\partial H(q,\dot q,p)}{\partial q^i} = -\frac{\partial L(q,\dot q)}{\partial q^i}\,, \qquad \frac{\partial H(q,\dot q,p)}{\partial \dot q^i} = 0\,, \qquad \frac{\partial H(q,\dot q,p)}{\partial p_i} = \dot q^i\,.$$

Therefore we have, for $(q,p) = \mathcal{L}(q,\dot q)$,

$$\frac{\partial \widehat H(q,p)}{\partial q^i} = \frac{\partial \widetilde H(q,\dot q,p)}{\partial q^i} = -\frac{\partial L(q,\dot q)}{\partial q^i} - \sum_{\alpha=1}^{r} v^\alpha(q,\dot q,t)\frac{\partial \widetilde\Phi_\alpha(q,\dot q,p)}{\partial q^i}\,,$$

$$\frac{\partial \widehat H(q,p)}{\partial p_i} = \frac{\partial \widetilde H(q,\dot q,p)}{\partial p_i} = \dot q^i - \sum_{\alpha=1}^{r} v^\alpha(q,\dot q,t)\frac{\partial \widetilde\Phi_\alpha(q,\dot q,p)}{\partial p_i}\,.$$

Generalized Hamiltonian dynamics (5)

Since $\widetilde{\Phi}_\alpha = \Phi_\alpha \circ \pi_{T^*Q}$, these equations become, for $(q,p) = \mathcal{L}(q,\dot{q})$,

$$\frac{\partial \widehat{H}(q,p)}{\partial q^i} = -\frac{\partial L(q,\dot{q})}{\partial q^i} - \sum_{\alpha=1}^{r} v^\alpha(q,\dot{q},t)\frac{\partial \Phi_\alpha(q,p)}{\partial q^i} \,,$$

$$\frac{\partial \widehat{H}(q,p)}{\partial p_i} = \dot{q}^i - \sum_{\alpha=1}^{r} v^\alpha(q,\dot{q},t)\frac{\partial \Phi_\alpha(q,p)}{\partial p_i} \,.$$

Generalized Hamiltonian dynamics (5)

Since $\widetilde{\Phi}_\alpha = \Phi_\alpha \circ \pi_{T^*Q}$, these equations become, for $(q,p) = \mathcal{L}(q,\dot q)$,

$$\frac{\partial \widehat{H}(q,p)}{\partial q^i} = -\frac{\partial L(q,\dot q)}{\partial q^i} - \sum_{\alpha=1}^{r} v^\alpha(q,\dot q,t)\frac{\partial \Phi_\alpha(q,p)}{\partial q^i}\,,$$

$$\frac{\partial \widehat{H}(q,p)}{\partial p_i} = \dot q^i - \sum_{\alpha=1}^{r} v^\alpha(q,\dot q,t)\frac{\partial \Phi_\alpha(q,p)}{\partial p_i}\,.$$

From the equations of motion in Lagrange's formalism

$$\frac{dq^i}{dt} = \dot q^i\,,\qquad \frac{d}{dt}\left(\frac{\partial L(q,\dot q)}{\partial \dot q^i}\right) = \frac{\partial L(q,\dot q)}{\partial q^i}$$

we deduce the equations of motion in the generalized Hamilton's formalism:

Generalized Hamiltonian dynamics (6)

$$\begin{cases} \dfrac{dq^i}{dt} = \dfrac{\partial \widehat{H}(q,p)}{\partial p_i} + \displaystyle\sum_{\alpha=1}^{r} v^\alpha(q,\dot{q},t)\dfrac{\partial \Phi_\alpha(q,p)}{\partial p_i}\,, \\ \dfrac{dp_i}{dt} = -\dfrac{\partial \widehat{H}(q,p)}{\partial q^i} + \displaystyle\sum_{\alpha=1}^{r} v^\alpha(q,\dot{q},t)\dfrac{\partial \Phi_\alpha(q,p)}{\partial q^i}\,. \end{cases}$$

Generalized Hamiltonian dynamics (6)

$$\begin{cases} \dfrac{dq^i}{dt} = \dfrac{\partial \widehat{H}(q,p)}{\partial p_i} + \displaystyle\sum_{\alpha=1}^{r} v^\alpha(q,\dot q,t)\dfrac{\partial \Phi_\alpha(q,p)}{\partial p_i}\,, \\[2ex] \dfrac{dp_i}{dt} = -\dfrac{\partial \widehat{H}(q,p)}{\partial q^i} + \displaystyle\sum_{\alpha=1}^{r} v^\alpha(q,\dot q,t)\dfrac{\partial \Phi_\alpha(q,p)}{\partial q^i}\,. \end{cases}$$

These equations, valid for $(q,p)=\mathcal{L}(q,\dot q)$, follow from Lagrange's equations. Under Dirac's assumptions ($\mathcal{L}$ of constant rank, existence of the functions $\widehat{H}$ and Φ_α), once these (not uniquely determined) functions are chosen, for each solution $t\mapsto q(t)$ of Lagrange's equations, $t\mapsto \bigl(q(t),\dot q(t),p(t)\bigr)$ is a solution of the above generalized Hamilton's equations (for a suitable choice of the $v^\alpha(q,\dot q,t)$, which may not be unique). We have set $\dot q(t)=\dfrac{dq(t)}{dt}$ and $\bigl(q(t),p(t)\bigr)=\mathcal{L}\bigl(q(t),\dot q(t)\bigr)$.

Secondary constraints (1)

The generalized Hamilton's equations

$$\begin{cases} \dfrac{dq^i}{dt} = \dfrac{\partial \widehat{H}(q,p)}{\partial p_i} + \displaystyle\sum_{\alpha=1}^{r} v^\alpha \dfrac{\partial \Phi_\alpha(q,p)}{\partial p_i}\,, \\ \dfrac{dp_i}{dt} = -\dfrac{\partial \widehat{H}(q,p)}{\partial q^i} + \displaystyle\sum_{\alpha=1}^{r} v^\alpha \dfrac{\partial \Phi_\alpha(q,p)}{\partial q^i} \end{cases}$$

can be considered as differential equations on T^*Q on ther own, the v^α being now considered as unknown functions of the time t. From that point of view, they make an *under-determined system*, since the v^α can be chosen arbitrarily.

Secondary constraints (2)

Using the Poisson bracket on T^*Q associated to its canonical symplectic structure, these equations can be written

$$\frac{dg}{dt} = \{\widehat{H}, g\} + \sum_{\alpha=1}^{r} v^{\alpha}\{\Phi_{\alpha}, g\}\,,$$

where g is any smooth function on T^*Q. Hamilton's generalized equations in local coordinates are obtained when g is one of the local coordinate functions q^i or p_i.

Secondary constraints (2)

Using the Poisson bracket on T^*Q associated to its canonical symplectic structure, these equations can be written

$$\frac{dg}{dt} = \{\widehat{H}, g\} + \sum_{\alpha=1}^{r} v^\alpha \{\Phi_\alpha, g\}\,,$$

where g is any smooth function on T^*Q. Hamilton's generalized equations in local coordinates are obtained when g is one of the local coordinate functions q^i or p_i.

To be an image, by the Legendre map $\mathcal{L}$, of a solution of Lagrange's equations, a solution $t \mapsto \bigl(q(t), p(t)\bigr)$ of the generalized Hamilton's equations must lie in $D_0 = \mathcal{L}(TQ)$. i.e., must satisfy

$$\Phi_\alpha\bigl(q(t), p(t)\bigr) = 0 \quad \text{for all } t\,, \quad 1 \leq \alpha \leq r\,.$$

Secondary constraints (3)

That necessary condition is satisfied if the starting point $\big(q(t_0), p(t_0)\big)$ of that solution is in D_0 and if the following *compatibility conditions* are satisfied:

$$\frac{d\Phi_\beta}{dt} = \{\widehat{H}, \Phi_\beta\} + \sum_{\alpha=1}^{r} v^\alpha \{\Phi_\alpha, \Phi_\beta\} = 0\,, \quad 1 \leq \beta \leq r\,.$$

When explicited, these equations may give rise to:

Secondary constraints (3)

That necessary condition is satisfied if the starting point $\big(q(t_0), p(t_0)\big)$ of that solution is in D_0 and if the following *compatibility conditions* are satisfied:

$$\frac{d\Phi_\beta}{dt} = \{\widehat{H}, \Phi_\beta\} + \sum_{\alpha=1}^{r} v^\alpha \{\Phi_\alpha, \Phi_\beta\} = 0\,, \quad 1 \leq \beta \leq r\,.$$

When explicited, these equations may give rise to:

- equalities satified if $\Phi_\alpha = 0$ for $1 \leq \alpha \leq r$;

Secondary constraints (3)

That necessary condition is satisfied if the starting point $\big(q(t_0), p(t_0)\big)$ of that solution is in D_0 and if the following *compatibility conditions* are satisfied:

$$\frac{d\Phi_\beta}{dt} = \{\widehat{H}, \Phi_\beta\} + \sum_{\alpha=1}^{r} v^\alpha \{\Phi_\alpha, \Phi_\beta\} = 0\,, \quad 1 \leq \beta \leq r\,.$$

When explicited, these equations may give rise to:

- equalities satified if $\Phi_\alpha = 0$ for $1 \leq \alpha \leq r$;
- impossible equalities such as $1 = 0$;

Secondary constraints (3)

That necessary condition is satisfied if the starting point $\big(q(t_0), p(t_0)\big)$ of that solution is in D_0 and if the following *compatibility conditions* are satisfied:

$$\frac{d\Phi_\beta}{dt} = \{\widehat{H}, \Phi_\beta\} + \sum_{\alpha=1}^{r} v^\alpha \{\Phi_\alpha, \Phi_\beta\} = 0\,, \quad 1 \leq \beta \leq r\,.$$

When explicited, these equations may give rise to:

- equalities satified if $\Phi_\alpha = 0$ for $1 \leq \alpha \leq r$;
- impossible equalities such as $1 = 0$;
- equations which restrict the generality of the v^α;

Secondary constraints (3)

That necessary condition is satisfied if the starting point $\big(q(t_0), p(t_0)\big)$ of that solution is in D_0 and if the following *compatibility conditions* are satisfied:

$$\frac{d\Phi_\beta}{dt} = \{\widehat{H}, \Phi_\beta\} + \sum_{\alpha=1}^{r} v^\alpha \{\Phi_\alpha, \Phi_\beta\} = 0\,, \quad 1 \leq \beta \leq r\,.$$

When explicited, these equations may give rise to:

- equalities satified if $\Phi_\alpha = 0$ for $1 \leq \alpha \leq r$;
- impossible equalities such as $1 = 0$;
- equations which restrict the generality of the v^α;
- new equalities, called *secondary constraints*

$$\chi_k(p, q) = 0\,, \quad k = 1\,,\ 2\,,\ \ldots\,.$$

the χ_k being smooth functions on open subsets of T^*Q.

Dirac's algorithm (1)

Unless impossible equalities occur (which means that the Lagrangian used is inconsistent), secondary constraints lead to new *compatibility conditions*

$$\frac{d\chi_k}{dt} = \{\widehat{H}, \chi_k\} + \sum_{\alpha=1}^{r} v^\alpha \{\Phi_\alpha, \chi_k\} = 0\,, \quad k = 1\,,\ 2\,,\ \ldots\,,$$

which again may give rise to satisfied equalities, impossible equalities, new relations involving the v^α and new secondary constraints.

Dirac's algorithm (1)

Unless impossible equalities occur (which means that the Lagrangian used is inconsistent), secondary constraints lead to new *compatibility conditions*

$$\frac{d\chi_k}{dt} = \{\widehat{H}, \chi_k\} + \sum_{\alpha=1}^{r} v^\alpha \{\Phi_\alpha, \chi_k\} = 0\,, \qquad k = 1\,,\ 2\,,\ \ldots\,,$$

which again may give rise to satisfied equalities, impossible equalities, new relations involving the v^α and new secondary constraints.

The process, called *Dirac's algorithm*, is pursued until no new constraints appear.

Dirac's algorithm (2)

Assuming that no impossible equation occured, we are left with a total of s *secondary constraints*

$$\chi_k(q,p) = 0\,, \quad 1 \leq k \leq s\,,$$

and, eventually, u nonhomogeneous linear equations, with functions on open subsets of T^*Q as coefficients, which must be satisfied by the v^α:

$$\sum_{\alpha=1}^{r} A^l_\alpha(q,p) v^\alpha = B^l(q,p)\,, \quad 1 \leq l \leq u\,.$$

Final constraint submanifold (1)

Finally we have a total of $r + s$ *constraints*

$$\Phi_\alpha(q,p) = 0\,,\ 1 \leq \alpha \leq r\,, \quad \chi_k(q,p) = 0\,,\ 1 \leq k \leq s\,,$$

which define a subset D_f of T^*Q, contained in the image D_0 of the Legendre map $\mathcal{L}$. The functions Φ_α and χ_k are called the *primary* and *secondary constraint functions*. We will assume that their differentials are linearly independent at each point in D_f. So D_f is a smooth $(2n - r - s)$-dimensional submanifold of T^*Q, called the *final constraint submanifold*, contained in the initial constraint submanifold D_0.

Final constraint submanifold (1)

Finally we have a total of $r + s$ *constraints*

$$\Phi_\alpha(q,p) = 0\,,\ 1 \le \alpha \le r\,, \quad \chi_k(q,p) = 0\,,\ 1 \le k \le s\,,$$

which define a subset D_f of T^*Q, contained in the image D_0 of the Legendre map $\mathcal{L}$. The functions Φ_α and χ_k are called the *primary* and *secondary constraint functions*. We will assume that their differentials are linearly independent at each point in D_f. So D_f is a smooth $(2n-r-s)$-dimensional submanifold of T^*Q, called the *final constraint submanifold*, contained in the initial constraint submanifold D_0.

Only the primary constraint functions Φ_α appear in the generalized Hamilton equations

$$\frac{dg}{dt} = \{\widehat{H}, g\} + \sum_{\alpha=1}^{r} v^\alpha \{\Phi_\alpha, g\}\,, \quad g \text{ any smooth function on } T^*Q\,.$$

Final constraint submanifold (2)

Geometrical Interpretation Let $X_{\widehat{H}}$ and X_{Φ_α} be the Hamiltonian vector fields on T^*Q with Hamiltonians $\widehat{H}$ and Φ_α, $(1 \leq \alpha \leq r)$, respectively.

Final constraint submanifold (2)

Geometrical Interpretation Let $X_{\widehat{H}}$ and X_{Φ_α} be the Hamiltonian vector fields on T^*Q with Hamiltonians $\widehat{H}$ and Φ_α, $(1 \leq \alpha \leq r)$, respectively.

Generally speaking, $X_{\widehat{H}}$ is not tangent to the initial constraint submanifold D_0. Dirac's algorithm solves the following problem:

Final constraint submanifold (2)

Geometrical Interpretation Let $X_{\widehat{H}}$ and X_{Φ_α} be the Hamiltonian vector fields on T^*Q with Hamiltonians $\widehat{H}$ and Φ_α, $(1 \leq \alpha \leq r)$, respectively.

Generally speaking, $X_{\widehat{H}}$ is not tangent to the initial constraint submanifold D_0. Dirac's algorithm solves the following problem:

Find a submanifold $D_f \subset D_0$ such that for each point z of that submanifold, there exist coefficients v^α for which the vector

$$X_{\widehat{H}}(z) + \sum_{\alpha=1}^{r} v^\alpha X_{\Phi_\alpha}(z)$$

is tangent to D_f at z.

Final constraint submanifold (3)

In other words, for each $z \in D_f$, *the intersection of* $T_z D_f$ *with the affine subspace of* $T_z(T^*Q)$ *made by the vectors* $X_{\widehat{H}}(z) + \sum_{\alpha=1}^{r} v^\alpha X_{\Phi_\alpha}(z)$*, for all reals* v^α*, must be not empty.*

Final constraint submanifold (3)

In other words, for each $z \in D_f$, *the intersection of* $T_z D_f$ *with the affine subspace of* $T_z(T^*Q)$ *made by the vectors* $X_{\widehat{H}}(z) + \sum_{\alpha=1}^{r} v^\alpha X_{\Phi_\alpha}(z)$*, for all reals* v^α*, must be not empty.*

That intersection is not always reduced to only one vector: *it may be an affine subspace of* $T_z D_f$.

Final constraint submanifold (3)

In other words, for each $z \in D_f$, *the intersection of* $T_z D_f$ *with the affine subspace of* $T_z(T^*Q)$ *made by the vectors* $X_{\widehat{H}}(z) + \sum_{\alpha=1}^{r} v^\alpha X_{\Phi_\alpha}(z)$*, for all reals* v^α*, must be not empty.*

That intersection is not always reduced to only one vector: *it may be an affine subspace of* $T_z D_f$.

For example, if one of the vectors $X_{\Phi_\alpha}(z)$ is tangent to D_f, the corresponding coefficient v^α can be any real number and that intersection contains an affine straight line parallel to that vector.

Final constraint submanifold (3)

In other words, for each $z \in D_f$, *the intersection of* $T_z D_f$ *with the affine subspace of* $T_z(T^*Q)$ *made by the vectors* $X_{\widehat{H}}(z) + \sum_{\alpha=1}^{r} v^\alpha X_{\Phi_\alpha}(z)$*, for all reals* v^α*, must be not empty.*

That intersection is not always reduced to only one vector: *it may be an affine subspace of* $T_z D_f$.

For example, if one of the vectors $X_{\Phi_\alpha}(z)$ is tangent to D_f, the corresponding coefficient v^α can be any real number and that intersection contains an affine straight line parallel to that vector.

These considerations lead Dirac to distnguish two kinds of constraints: *first class constraints* and *second class constraints*.

Final constraint submanifold (3)

In other words, for each $z \in D_f$, *the intersection of* $T_z D_f$ *with the affine subspace of* $T_z(T^*Q)$ *made by the vectors* $X_{\widehat{H}}(z) + \sum_{\alpha=1}^{r} v^\alpha X_{\Phi_\alpha}(z)$, *for all reals* v^α, *must be not empty.*

That intersection is not always reduced to only one vector: *it may be an affine subspace of* $T_z D_f$.

For example, if one of the vectors $X_{\Phi_\alpha}(z)$ is tangent to D_f, the corresponding coefficient v^α can be any real number and that intersection contains an affine straight line parallel to that vector.

These considerations lead Dirac to distnguish two kinds of constraints: *first class constraints* and *second class constraints*.

The distinction between *first class* and *second class* constraints is independent of the distinction between *primary* and *secondary* constraints.

First and second class constraints (1)

Definition A smooth function on T^*Q is said to be *first class* if its Poisson bracket with the constraint functions (primary as well as secondary) Φ_α and χ_k vanishes on D_f, $1 \leq \alpha \leq r$, $1 \leq k \leq s$.

A function which is not first class is said to be *second class*.

First and second class constraints (1)

Definition A smooth function on T^*Q is said to be *first class* if its Poisson bracket with the constraint functions (primary as well as secondary) Φ_α and χ_k vanishes on D_f, $1 \leq \alpha \leq r$, $1 \leq k \leq s$.

A function which is not first class is said to be *second class*.

These definitions apply to the constraint functions Φ_α and χ_k themselves. So we distinguish between *first class* and *second class constraint functions*.

First and second class constraints (1)

Definition A smooth function on T^*Q is said to be *first class* if its Poisson bracket with the constraint functions (primary as well as secondary) Φ_α and χ_k vanishes on D_f, $1 \leq \alpha \leq r$, $1 \leq k \leq s$.

A function which is not first class is said to be *second class*.

These definitions apply to the constraint functions Φ_α and χ_k themselves. So we distinguish between *first class* and *second class constraint functions*.

The Poisson bracket of two first class functions is first class (Jacobi identity). Any linear combination of first class functions, with functions defined on T^*Q as coefficients, is first class.

First and second class constraints (1)

Definition A smooth function on T^*Q is said to be *first class* if its Poisson bracket with the constraint functions (primary as well as secondary) Φ_α and χ_k vanishes on D_f, $1 \leq \alpha \leq r$, $1 \leq k \leq s$.

A function which is not first class is said to be *second class*.

These definitions apply to the constraint functions Φ_α and χ_k themselves. So we distinguish between *first class* and *second class constraint functions*.

The Poisson bracket of two first class functions is first class (Jacobi identity). Any linear combination of first class functions, with functions defined on T^*Q as coefficients, is first class.

Geometrical interpretation . A smooth function Ψ is first class if and only if the Hamiltonian vector field X_Ψ is everywhere tangent to the final constraint submanifold D_f

First and second class constraints (2)

When the constraint function Ψ is chosen among the Φ_α's and χ_k's obtained with Dirac's algorithm, the correpondıng Hamiltonan vector field X_ψ may be tangent to the final constraint submanifold D_f at some points, but not everywhere. For that reason, instead of the original constraint functions Φ_α and χ_k, Dirac uses a set of $r+s$ new constraint functions Ψ_γ, $1 \leq \gamma \leq r+s$, obtained from the original ones by a linear transformation whose coefficients are functions on T^*Q, the determinant of that linear transformation being nowhere zero. The submanifold $D_f \subset T^*Q$ can now be defined by the equations

$$\Theta_\gamma(q,p) = 0\,, \quad 1 \leq \gamma \leq r+s\,,$$

instead of $\Psi_\alpha(q,p) = 0$ and $\chi_k(q,p) = 0$, $1 \leq \alpha \leq r$, $1 \leq k \leq s$.

Poisson brackets of second class constraints

Dirac chooses the transformation which yields the Θ_γ's as linear combinations of the Φ_α's and the χ_k's, in such a way that the number of first class Θ_γ's is the largest possible. By reordering, we may assume that the Θ_γ are:

first class for $1 \leq \gamma \leq k\,,$ *second class for* $k+1 \leq \gamma \leq r+s\,.$

Poisson brackets of second class constraint

Dirac chooses the transformation which yields the Θ_γ's as linear combinations of the Φ_α's and the χ_k's, in such a way that the number of first class Θ_γ's is the largest possible. By reordering, we may assume that the Θ_γ are:

first class for $1 \leq \gamma \leq k\,,$ *second class for* $k+1 \leq \gamma \leq r+s\,.$

Dirac proves that the matrix whose coefficients are the Poisson brackets of pairs of second class Θ_γ's,

$$\big(\{\Theta_\alpha, \Theta_\beta\}\big)\,, \quad k+1 \leq \alpha,\, \beta \leq r+s$$

is invertible at each point of D_f. The Poisson braket being skew-symmetric, it implies that the number $r+s-k$ of second class constraints is even. We will set $r+s-k = 2p$. This result has the following symplectic explanation.

Symplectic explanation (1)

For each point $z \in D_f$ the tangent space $T_z D_f$ is the annihilator of the vector subspace of $T_z^*(T^*Q)$ generated by the $d\Theta_\gamma(z)$, $1 \leq \gamma \leq r+s$. Equipped with $\omega(z)$, $T_z(T^*Q)$ is a symplectic vector space. The symplectic orthogonal $\mathrm{orth}(T_z D_f)$ of its vector subspace $T_z D_f$ is the vector subspace of $T_z(T^*Q)$ generated by the Hamiltonian vectors $X_{\Theta_\gamma}(z)$, $1 \leq \gamma \leq r+s$.

Symplectic explanation (1)

For each point $z \in D_f$ the tangent space $T_z D_f$ is the annihilator of the vector subspace of $T_z^*(T^*Q)$ generated by the $d\Theta_\gamma(z)$, $1 \leq \gamma \leq r+s$. Equipped with $\omega(z)$, $T_z(T^*Q)$ is a symplectic vector space. The symplectic orthogonal $\mathrm{orth}(T_z D_f)$ of its vector subspace $T_z D_f$ is the vector subspace of $T_z(T^*Q)$ generated by the Hamiltonian vectors $X_{\Theta_\gamma}(z)$, $1 \leq \gamma \leq r+s$.

When Θ_γ is first class, $X_{\Theta_\gamma}(z) \in T_z D_f \cap \mathrm{orth}(T_z D_f)$. This subspace is the common kernel of the 2-forms induced by $\omega(z)$ both on the vector subspace $T_z D_f$ and on its symplectic orthogonal $\mathrm{orth}(T_z D_f)$.

Symplectic explanation (1)

For each point $z \in D_f$ the tangent space $T_z D_f$ is the annihilator of the vector subspace of $T^*_z(T^*Q)$ generated by the $d\Theta_\gamma(z)$, $1 \leq \gamma \leq r+s$. Equipped with $\omega(z)$, $T_z(T^*Q)$ is a symplectic vector space. The symplectic orthogonal $\mathrm{orth}(T_z D_f)$ of its vector subspace $T_z D_f$ is the vector subspace of $T_z(T^*Q)$ generated by the Hamiltonian vectors $X_{\Theta_\gamma}(z)$, $1 \leq \gamma \leq r+s$.

When Θ_γ is first class, $X_{\Theta_\gamma}(z) \in T_z D_f \cap \mathrm{orth}(T_z D_f)$. This subspace is the common kernel of the 2-forms induced by $\omega(z)$ both on the vector subspace $T_z D_f$ and on its symplectic orthogonal $\mathrm{orth}(T_z D_f)$.

It seems that *Dirac tacitly assumes that the 2-form induced on D_f by ω is of constant rank*. I tried (without succes) to prove that the constancy of rank followed from Dirac's algorithm.

Symplectic explanation (2)

Under this assumption, we can indeed *split (at least locally) the constraint functions Θ_γ into first and second class*. At each point $z \in D_f$, the Hamiltonian vectors $X_{\Theta_\gamma}(z)$, with Θ_γ second class, form a basis of a symplectic vector subspace of $T_z(T^*Q)$. That explains why the matrix of Poisson brackets of secondary constraint functions is invertible at each point $z \in D_f$.

Symplectic explanation (2)

Under this assumption, we can indeed *split (at least locally) the constraint functions Θ_γ into first and second class*. At each point $z \in D_f$, the Hamiltonian vectors $X_{\Theta_\gamma}(z)$, with Θ_γ second class, form a basis of a symplectic vector subspace of $T_z(T^*Q)$. That explains why the matrix of Poisson brackets of secondary constraint functions is invertible at each point $z \in D_f$.

In what follows we will denote by Θ_γ $(1 \leq \gamma \leq k)$ the *first class constraint functions* and by $\Psi_\delta = \Theta_{k+\delta}$ $(1 \leq \delta \leq 2p)$ the *second class constraint functions*.

The Dirac bracket (1)

Dirac introduces, for functions defined on T^*Q, a modified Poisson bracket (we will call it the *Dirac bracket* and denote it by $\{\ ,\ \}_D$) with the following properties:

The Dirac bracket (1)

Dirac introduces, for functions defined on T^*Q, a modified Poisson bracket (we will call it the *Dirac bracket* and denote it by $\{\ ,\ \}_D$) with the following properties:

- that modified Poisson bracket is defined on the open neighbourhood W of D_f in T^*Q on which the matrix with coefficients $\{\Psi_\gamma, \Psi_\delta\}$ is invertible);

The Dirac bracket (1)

Dirac introduces, for functions defined on T^*Q, a modified Poisson bracket (we will call it the *Dirac bracket* and denote it by $\{\ ,\ \}_D$) with the following properties:

- that modified Poisson bracket is defined on the open neighbourhood W of D_f in T^*Q on which the matrix with coefficients $\{\Psi_\gamma, \Psi_\delta\}$ is invertible);
- the Dirac bracket $\{\Psi_\gamma, g\}_D$ of a second class contraint function Ψ_γ ($1 \leq \gamma \leq 2p$) with any smooth function g vanishes identically on W;

The Dirac bracket (1)

Dirac introduces, for functions defined on T^*Q, a modified Poisson bracket (we will call it the *Dirac bracket* and denote it by $\{\ ,\ \}_D$) with the following properties:

- that modified Poisson bracket is defined on the open neighbourhood W of D_f in T^*Q on which the matrix with coefficients $\{\Psi_\gamma, \Psi_\delta\}$ is invertible);
- the Dirac bracket $\{\Psi_\gamma, g\}_D$ of a second class contraint function Ψ_γ ($1 \leq \gamma \leq 2p$) with any smooth function g vanishes identically on W;
- When f and g are two smooth functions on T^*Q and $z \in W$ a point at which $\{f, \Psi_\gamma\} = 0$ and $\{g, \Psi_\gamma\} = 0$, $1 \leq \gamma \leq 2p$, then

$$\{f, g\}_D(z) = \{f, g\}(z)\,.$$

The Dirac bracket (2)

We have seen that the matrix with coefficients

$$M_{\alpha\beta} = \{\Psi_\alpha, \Psi_\beta\}, \quad 1 \leq \alpha, \beta \leq 2p$$

is invertible.

The Dirac bracket (2)

We have seen that the matrix with coefficients

$$M_{\alpha\beta} = \{\Psi_\alpha, \Psi_\beta\}, \quad 1 \leq \alpha, \beta \leq 2p$$

is invertible. Let $C_{\alpha\beta}$ be the coefficients of its inverse. They are smooth functions on W such that

$$\sum_{\beta=1}^{2p} M_{\alpha\beta} C_{\beta\gamma} = \delta_{\alpha\gamma} .$$

The Dirac bracket (2)

We have seen that the matrix with coefficients

$$M_{\alpha\beta} = \{\Psi_\alpha, \Psi_\beta\}\,, \quad 1 \le \alpha, \beta \le 2p$$

is invertible. Let $C_{\alpha\beta}$ be the coefficients of its inverse. They are smooth functions on W such that

$$\sum_{\beta=1}^{2p} M_{\alpha\beta} C_{\beta\gamma} = \delta_{\alpha\gamma}\,.$$

The Dirac bracket of two smooth functions f and g is defined as

$$\{f, g\}_D = \{f, g\} - \sum_{\alpha=1}^{2p} \sum_{\beta=1}^{2p} \{f, \Psi_\alpha\} C_{\alpha\beta} \{\Psi_\beta, g\}\,.$$

The Dirac bracket (3)

In [1] Dirac proves, by direct calculations, that his bracket has all the properties of a Poisson bracket (*skew-symmetry*, *Leibniz identity* and *Jacobi identity*).

The Dirac bracket (3)

In [1] Dirac proves, by direct calculations, that his bracket has all the properties of a Poisson bracket (*skew-symmetry*, *Leibniz identity* and *Jacobi identity*).

On the final constraint submanifold D_f, the constraint functions are equal to 0, therefore the generalized Hamilton equations can be written, g being any smooth function

$$\frac{dg}{dt} = \{\widehat{H} + \sum_{\alpha=1}^{r} v^{\alpha}\Phi_{\alpha}, g\} = \{H_T, g\}\,.$$

The Dirac bracket (3)

In [1] Dirac proves, by direct calculations, that his bracket has all the properties of a Poisson bracket (*skew-symmetry*, *Leibniz identity* and *Jacobi identity*).

On the final constraint submanifold D_f, the constraint functions are equal to 0, therefore the generalized Hamilton equations can be written, g being any smooth function

$$\frac{dg}{dt} = \{\widehat{H} + \sum_{\alpha=1}^{r} v^\alpha \Phi_\alpha, g\} = \{H_T, g\}\,.$$

The function $H_T = \widehat{H} + \sum_{\alpha=1}^{r} v^\alpha \Phi_\alpha$ is first class, so the generalized Hamilton equation can be written with the Dirac bracket, as well as with the ordinary bracket

$$\frac{dg}{dt} = \{H_T, g\}_D\,.$$

The Poisson-Dirac bivector (1)

The usual Poisson bracket is built with the bivector field Λ on T^*Q, given in Darboux coordibates by $\Lambda = \sum_{i=1}^{n} \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q^i}$.

Let $\Lambda^\sharp : T^*Q \to TQ$ be the bundle morphism determined by Λ. For smooth functions f and g,

$$\{f, g\} = \Lambda(df, dg) = i(X_f)dg\,, \quad \text{with } X_f = \Lambda^\sharp(df)\,.$$

The Poisson-Dirac bivector (1)

The usual Poisson bracket is built with the bivector field Λ on T^*Q, given in Darboux coordibates by $\Lambda = \sum_{i=1}^{n} \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q^i}$.

Let $\Lambda^\sharp : T^*Q \to TQ$ be the bundle morphism determined by Λ. For smooth functions f and g,

$$\{f, g\} = \Lambda(df, dg) = i(X_f)dg \,, \quad \text{with } X_f = \Lambda^\sharp(df)\,.$$

Let Λ_D be the bivector field corresponding to $\{\ ,\ \}_D$.

The Poisson-Dirac bivector (1)

The usual Poisson bracket is built with the bivector field Λ on T^*Q, given in Darboux coordibates by $\Lambda = \sum_{i=1}^{n} \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q^i}$.

Let $\Lambda^\sharp : T^*Q \to TQ$ be the bundle morphism determined by Λ. For smooth functions f and g,

$$\{f,g\} = \Lambda(df,dg) = i(X_f)dg\,, \quad \text{with } X_f = \Lambda^\sharp(df)\,.$$

Let Λ_D be the bivector field corresponding to $\{\ ,\ \}_D$.

Let F be the rank $2p$ vector subbundle of $T(T^*Q)$ generated by the Hamiltonian vector fields X_{Ψ_α}, $1 \leq \alpha \leq 2p$, and $G = \operatorname{orth} F$ be its symplectic orthogonal. Both F and G are symplectic vector subbundles and

$$T(T^*Q) = F \oplus \operatorname{orth} F = F \oplus G\,.$$

The Poisson-Dirac bivector (2)

By duality $T^*(T^*Q) = F^0 \oplus G^0$, where F^0 and G^0 are the annihilators of F and G, respectively. Let $\pi_{F^0} : T^*(T^*Q) \to F^0$ and $\pi_{G^0} : T^*(T^*Q) \to G^0$ be the projections with kernels G^0 and F^0, respectively.

The Poisson-Dirac bivector (2)

By duality $T^*(T^*Q) = F^0 \oplus G^0$, where F^0 and G^0 are the annihilators of F and G, respectively. Let $\pi_{F^0} : T^*(T^*Q) \to F^0$ and $\pi_{G^0} : T^*(T^*Q) \to G^0$ be the projections with kernels G^0 and F^0, respectively.

We have, for $z \in T^*Q$, η and $\zeta \in T^*(T^*Q)$,

$$\Lambda_D(\eta, \zeta) = \Lambda\Big(\pi_{F^0}(\eta), \pi_{F^0}(\zeta)\Big) .$$

The Poisson-Dirac bivector (2)

By duality $T^*(T^*Q) = F^0 \oplus G^0$, where F^0 and G^0 are the annihilators of F and G, respectively. Let $\pi_{F^0} : T^*(T^*Q) \to F^0$ and $\pi_{G^0} : T^*(T^*Q) \to G^0$ be the projections with kernels G^0 and F^0, respectively.

We have, for $z \in T^*Q$, η and $\zeta \in T^*(T^*Q)$,

$$\Lambda_D(\eta, \zeta) = \Lambda\bigl(\pi_{F^0}(\eta), \pi_{F^0}(\zeta)\bigr) \, .$$

Therefore

$$\Lambda_D^\sharp = {}^t\pi_{F^0} \circ \Lambda^\sharp \circ \pi_{F^0} \, ,$$

where the transpose ${}^t\pi_{F^0} : G \to T(T^*Q)$ of $\pi_{F^0} : T^*(T^*Q) \to F^0$ is the canonical injection (F^0 being identified with the dual of G)

The Poisson-Dirac bivector (3)

Dirac has proven that Λ_D is a Poisson bivector. That property is a special case of the following proposition.

The Poisson-Dirac bivector (3)

Dirac has proven that Λ_D is a Poisson bivector. That property is a special case of the following proposition.

Proposition Let (M, ω) be a symplectic manifold, F a symplectic vector subbundle of TM and $G = \operatorname{orth} F$ its symplectic orthogonal. Let

$$\Lambda_D^\sharp = {}^t\pi_{F^0} \circ \Lambda^\sharp \circ \pi_{F^0} \, ,$$

where Λ is the Poisson bivector associated to ω, π_{F^0} defined as above. Then Λ_D is a Poisson bivector field if and only if G is completely integrable.

The Poisson-Dirac bivector (3)

Dirac has proven that Λ_D is a Poisson bivector. That property is a special case of the following proposition.

Proposition Let (M, ω) be a symplectic manifold, F a symplectic vector subbundle of TM and $G = \operatorname{orth} F$ its symplectic orthogonal. Let

$$\Lambda_D^\sharp = {}^t\pi_{F^0} \circ \Lambda^\sharp \circ \pi_{F^0} \,,$$

where Λ is the Poisson bivector associated to ω, π_{F^0} defined as above. Then Λ_D is a Poisson bivector field if and only if G is completely integrable.

Proof If (M, Λ_D) is a Poisson manifold, G is the vector subbundle tangent to its symplectic leaves. So it is completely integrable.

The Poisson-Dirac bivector (3)

Dirac has proven that Λ_D is a Poisson bivector. That property is a special case of the following proposition.

Proposition Let (M,ω) be a symplectic manifold, F a symplectic vector subbundle of TM and $G = \mathrm{orth}\, F$ its symplectic orthogonal. Let

$$\Lambda_D^\sharp = {}^t\pi_{F^0} \circ \Lambda^\sharp \circ \pi_{F^0} \,,$$

where Λ is the Poisson bivector associated to ω, π_{F^0} defined as above. Then Λ_D is a Poisson bivector field if and only if G is completely integrable.

Proof If (M,Λ_D) is a Poisson manifold, G is the vector subbundle tangent to its symplectic leaves. So it is completely integrable.

Conversely, assume that G is completely integrable.

The Poisson-Dirac bivector (4)

We recall that if η and ζ are two 1-forms on M,

$$\Lambda_D(\eta,\zeta) = \Lambda\bigl(\pi_{F^0}(\eta), \pi_{F^0}(\zeta)\bigr) .$$

Let $\tau_M : TM \to M$ be the canonical projection of the tangent bundle. Then $(G, \tau_M|_G, M)$ is a Lie algebroid (with the bracket of vector fields, restricted to sections of $\tau_M|_G$ as composition law). Therefore the total space G^* of its dual bundle has a linear Poisson structure. But we know that G^* can be isentified with F^0, so F^0 has a linear Poisson structure. It means that the bracket (calculated for the Poisson structure Λ) of two 1-forms η and ζ on M which are sections of F^0 is again a section of F^0. The above formula for $\Lambda_D(\eta,\zeta)$ shows that Λ_D is a Poisson bivector field. $\square$

Compatibility of Λ and Λ_D

Under the assumptions of the above Proposition, we have two Poisson structures Λ and Λ_D on M. Generally speaking, *these two Poisson structures are not compatible*.

Compatibility of Λ and Λ_D

Under the assumptions of the above Proposition, we have two Poisson structures Λ and Λ_D on M. Generally speaking, *these two Poisson structures are not compatible*.

We have

$$\Lambda^\sharp - \Lambda_D^\sharp = {}^t\pi_{G^0} \circ \Lambda^\sharp \circ \pi_{G^0} \, ,$$

and the same proposition shows that $\Lambda - \Lambda_D$ is a Poisson bivector if and only if F is completely integrable.

Compatibility of Λ and Λ_D

Under the assumptions of the above Proposition, we have two Poisson structures Λ and Λ_D on M. Generally speaking, *these two Poisson structures are not compatible*.

We have

$$\Lambda^\sharp - \Lambda_D^\sharp = {}^t\pi_{G^0} \circ \Lambda^\sharp \circ \pi_{G^0} \,,$$

and the same proposition shows that $\Lambda - \Lambda_D$ is a Poisson bivector if and only if F is completely integrable.

When both G and F are completely integrable the manifold M is locally a product of two symplectic manifolds.

Example (1)

On $T\mathbb{R}^n$, with coordinates $(q^i, \dot{q}^i)$, $1 \leq i \leq n$, we take as Lagrangian

$$L_0(q, \dot{q}) = \frac{m}{2} \sum_{i=1}^{n} (\dot{q}^i)^2 - V(q)\,,$$

and we impose the constraint

$$F(q) = \text{constant}\,.$$

We add one dimension to the configuration manifold (coordinate λ). So we have two more dimensions on $T(\mathbb{R}^n \times \mathbb{R})$, with coordinates $(\lambda, \dot{\lambda})$. Our new Lagrangian is

$$L(q, \lambda, \dot{q}, \dot{\lambda}) = L_0(q, \dot{q}) + \dot{\lambda} F(q) = \frac{m}{2} \sum_{i=1}^{n} (\dot{q}^i)^2 - V(q) + \dot{\lambda} F(q)\,.$$

Example (2)

The Lagrange equations

$$\begin{cases} \dfrac{d}{dt}(m\dot{q}^i) + \dfrac{\partial V(q)}{\partial q^i} - \dot{\lambda}\dfrac{\partial F(q)}{\partial q^i} = 0\,, \\ \dfrac{d}{dt}F(q) = 0\,, \end{cases}$$

are the correct equations of motion of a heavy point constrained, by an ideal contraint, on a surface $F(q) =$ constant, that constant depending on the initial condition. The Lagrange multiplier is $\dot{\lambda}$.

Example (2)

The Lagrange equations

$$\begin{cases} \dfrac{d}{dt}(m\dot{q}^i) + \dfrac{\partial V(q)}{\partial q^i} - \dot{\lambda}\dfrac{\partial F(q)}{\partial q^i} = 0\,, \\ \dfrac{d}{dt}F(q) = 0\,, \end{cases}$$

are the correct equations of motion of a heavy point constrained, by an ideal contraint, on a surface $F(q) =$ constant, that constant depending on the initial condition.

The Lagrange multiplier is $\dot{\lambda}$.

The Legendre map is

$$\mathcal{L} : (q, \lambda, \dot{q}, \dot{\lambda}) \mapsto (q, \lambda, p, p_\lambda)\,, \quad \text{with}$$

$$p_i = m\dot{q}^i\,, \quad p_\lambda = F(q)\,.$$

Example (3)

The Hamiltonian $H(q, \lambda, \dot{q}, \dot{\lambda}, p, p_\lambda)$, defined on $T\mathbb{R}^{n+1} \oplus T^*\mathbb{R}^{n+1}$, is

$$H(q, \lambda, \dot{q}, \dot{\lambda}, p, p_\lambda) = \sum_{i=1}^{n} \Big(p_i - \frac{m}{2}\dot{q}^i\Big)\dot{q}^i + \dot{\lambda}\Big(p_\lambda - F(q)\Big) + V(q) \, .$$

Example (3)

The Hamiltonian $H(q,\lambda,\dot q,\dot\lambda,p,p_\lambda)$, defined on $T\mathbb{R}^{n+1}\oplus T^*\mathbb{R}^{n+1}$, is

$$H(q,\lambda,\dot q,\dot\lambda,p,p_\lambda)=\sum_{i=1}^{n}\big(p_i-\frac{m}{2}\dot q^i\big)\dot q^i+\dot\lambda\big(p_\lambda-F(q)\big)+V(q)\,.$$

As (non unique) Hamiltonian defined on $T^*\mathbb{R}^{n+1}$, we choose

$$\widehat{H}(q,\lambda,p,p_\lambda)=\frac{1}{2m}\sum_{i=1}^{n}p_i^2+V(q)\,.$$

Example (3)

The Hamiltonian $H(q, \lambda, \dot{q}, \dot{\lambda}, p, p_\lambda)$, defined on $T\mathbb{R}^{n+1} \oplus T^*\mathbb{R}^{n+1}$, is

$$H(q, \lambda, \dot{q}, \dot{\lambda}, p, p_\lambda) = \sum_{i=1}^{n} \big(p_i - \frac{m}{2}\dot{q}^i\big)\dot{q}^i + \dot{\lambda}\big(p_\lambda - F(q)\big) + V(q)\,.$$

As (non unique) Hamiltonian defined on $T^*\mathbb{R}^{n+1}$, we choose

$$\widehat{H}(q, \lambda, p, p_\lambda) = \frac{1}{2m}\sum_{i=1}^{n} p_i^2 + V(q)\,.$$

We have only one *primary constraint*

$$\Phi(q, \lambda, p, p_\lambda) = F(q) - p_\lambda = 0\,,$$

with constraint function $\Phi = F(q) - p_\lambda$.

Example (4)

The generalized Hamilton's equation for the time derivative of any smooth function g is

$$\frac{dg}{dt} = \{\widehat{H}, g\} + v\{\Phi, g\}\,.$$

Example (4)

The generalized Hamilton's equation for the time derivative of any smooth function g is

$$\frac{dg}{dt} = \{\widehat{H}, g\} + v\{\Phi, g\}\,.$$

The compatibility condition $\dfrac{d\Phi}{dt} = 0$ yields

$$\frac{1}{m}\chi(q, \lambda, p, p_\lambda) = \frac{1}{m}\sum_{i=1}^{n} p_i \frac{\partial F(q)}{\partial q^i} = 0\,.$$

Example (4)

The generalized Hamilton's equation for the time derivative of any smooth function g is

$$\frac{dg}{dt} = \{\widehat{H}, g\} + v\{\Phi, g\} \, .$$

The compatibility condition $\dfrac{d\Phi}{dt} = 0$ yields

$$\frac{1}{m}\chi(q, \lambda, p, p_\lambda) = \frac{1}{m}\sum_{i=1}^{n} p_i \frac{\partial F(q)}{\partial q^i} = 0 \, .$$

We obtain a *secondary constraint* $\chi = 0$, with constraint function

$$\chi = \sum_{i=1}^{n} p_i \frac{\partial F(q)}{\partial q^i} \, .$$

Example (5)

So we get another compatibility condition $\frac{d\chi}{dt} = 0$, which leads to

$$\sum_{i=1}^{n}\left(\frac{\partial F(q)}{\partial q^i}\right)^2 v = \sum_{i=1}^{n}\sum_{j=1}^{n} p_i p_j \frac{\partial^2 F(q)}{\partial q^i \partial q^j} - \sum_{i=1}^{n} \frac{\partial F(q)}{\partial q^i}\frac{\partial V(q)}{\partial q^i}\,.$$

Example (5)

So we get another compatibility condition $\frac{d\chi}{dt} = 0$, which leads to

$$\sum_{i=1}^{n} \left(\frac{\partial F(q)}{\partial q^i} \right)^2 v = \sum_{i=1}^{n} \sum_{j=1}^{n} p_i p_j \frac{\partial^2 F(q)}{\partial q^i \partial q^j} - \sum_{i=1}^{n} \frac{\partial F(q)}{\partial q^i} \frac{\partial V(q)}{\partial q^i} \,.$$

This equality is not a new compatibility condition; it is a relation which determines v as a function of $(q, \lambda, p, p_\lambda)$. We see that in fact v does not depend on λ nor on p_λ.

Example (5)

So we get another compatibility condition $\frac{d\chi}{dt} = 0$, which leads to

$$\sum_{i=1}^{n} \left(\frac{\partial F(q)}{\partial q^i} \right)^2 v = \sum_{i=1}^{n} \sum_{j=1}^{n} p_i p_j \frac{\partial^2 F(q)}{\partial q^i \partial q^j} - \sum_{i=1}^{n} \frac{\partial F(q)}{\partial q^i} \frac{\partial V(q)}{\partial q^i} \, .$$

This equality is not a new compatibility condition; it is a relation which determines v as a function of $(q, \lambda, p, p_\lambda)$. We see that in fact v does not depend on λ nor on p_λ.

So we have a total of two constraint functions: Φ and χ, with

$$\{\Phi, \chi\} = -\sum_{i=1}^{n} \left(\frac{\partial F(q)}{\partial q^i} \right)^2 \, .$$

The constraints $\Phi = 0$ and $\chi = 0$ are second class.

Example (6)

The generalized Hamilton equations for the coordinates functions are

$$\begin{cases} \dfrac{dq^i}{dt} = \dfrac{p_i}{m}\,, \\ \dfrac{dp_i}{dt} = -\dfrac{\partial V(q)}{\partial q^i} - v(q,p)\dfrac{\partial F(q)}{\partial q^i}\,, \end{cases} \qquad \begin{cases} \dfrac{d\lambda}{dt} = -v(q,p)\,, \\ \dfrac{dp_\lambda}{dt} = 0\,. \end{cases}$$

Example (6)

The generalized Hamilton equations for the coordinates functions are

$$\begin{cases} \dfrac{dq^i}{dt} = \dfrac{p_i}{m}\,, \\ \dfrac{dp_i}{dt} = -\dfrac{\partial V(q)}{\partial q^i} - v(q,p)\dfrac{\partial F(q)}{\partial q^i}\,, \end{cases} \qquad \begin{cases} \dfrac{d\lambda}{dt} = -v(q,p)\,, \\ \dfrac{dp_\lambda}{dt} = 0\,. \end{cases}$$

Remark Instead of $L(q,\lambda,\dot q,\dot\lambda) = L_0(q,\dot q) + \dot\lambda F(q)$, we may use as Lagrangian

$$L(q,\lambda,\dot q,\dot\lambda) = L_0(q,\dot q) + \lambda\big(F(q) - C\big)\,,$$

where C is a constant. We obtain the same equations of motion on the constraint manifold $F(q) = C$, and a similar generalized Hamiltonian formalism, with three constraints (one first class and two second class).

Example (7)

The Dirac brackets of the coordinates functions are

$$\{q^i, q^j\}_D = 0\,, \qquad \{q^i, \lambda\}_D = -\frac{1}{\{\Phi, \chi\}}\frac{\partial F}{\partial q^i}\,,$$

$$\{q^i, p_j\}_D = -\delta^i_j - \frac{1}{\{\Phi, \chi\}}\frac{\partial F}{\partial q^i}\frac{\partial F}{\partial q^j}\,, \qquad \{q^i, p_\lambda\}_D = 0\,,$$

$$\{p_i, \lambda\}_D = \frac{1}{\{\Phi, \chi\}}\sum_{k=1}^{n} p_k \frac{\partial^2 F}{\partial q^k \partial q^i}\,,$$

$$\{p_i, p_j\}_D = \frac{1}{\{\Phi, \chi\}}\sum_{k=1}^{n} p_k \left(\frac{\partial F}{\partial q^j}\frac{\partial^2 F}{\partial q^k \partial q^i} - \frac{\partial F}{\partial q^i}\frac{\partial^2 F}{\partial q^k \partial q^j}\right)\,,$$

$$\{p_i, p_\lambda\}_D = 0\,, \qquad \{p_\lambda, \lambda\} = 1\,.$$

Thanks

I thank the organizers of the conference *Thirty years of bihamiltonian systems*, Professor Maciej Blaszak and Professor Andriy Panasyuk, for giving me the opportunity to present this talk and to participate in that meeting.

And I thank the persons who had the kindness and patience for listening to my talk.

References (1)

[1] Paul A.M. Dirac, *Generalized Hamiltonian dynamics*, Canadian Journal of Mathematics, 2, 1950, 129–148.

[2] Paul A.M. Dirac, *Generalized Hamiltonian dynamics*, Proc. Roy. Soc. London A, 246, 1958, 326–332.

[3] Paul A.M. Dirac, *Lectures on Quantum Mechanics*, Belfer Graduate School of Science, Yeshiva University, New York, 1964.

[4] Mark J. Gotay, James M. Nester and George Hinds, *Presymplectic manifolds and the Dirac-Bergmann theory of constraints*, J. Math. Phys. 19 (11), 1978, 2388–2399.

[5] Maria Rosa Menzio and W.M. Tulczyjew, *Infinitesimal symplectic relations and generalized Hamiltonian dynamics*, Ann. Inst. Henri Poincaré, vol. XXVIII, 4 (1978), 349–367.

[6] D.C. Salisbury, *Peter Bergmann and the invention of constrained Hamiltonian dynamics*, July 23, 2006.

References (2)

[7] Jedrzej Śniatycki, *Dirac brackets in geometric dynamics*, Ann. Inst. Henri Poincaré, vol. XX, 4 (1974), 365–372.

[8] Ray Skinner and Raymond Rusk, *Generalized Hamiltonian dynamics. I. Formulation on $T^*Q \oplus TQ$*, J. Math. Phys. 24 (11), (1983), 2589–2594.

[9] Ray Skinner and Raymond Rusk, *Generalized Hamiltonian dynamics. II. Gauge transformations*, J. Math. Phys. 24 (11), (1983), 2595–2601.

[10] W.M. Tulczyjew, *Hamiltonian systems, Lagrangian systems and the Legendre transformation*, Symposia Mathematica XIV, 1974, 247–258.

[11] W.M. Tulczyjew, *The Legendre transformation*, Ann. Inst. Henri Poincaré, vol. XXVII, 1 (1977), 101–114.