On Jacobi manifolds and Jacobi bundles

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Abstract

We introduce the notion of a Jacobi bundle, which generalizes that of a Jacobi manifold. The construction of a Jacobi bundle over a conformal Jacobi manifold has, as particular cases, the constructions made by A. Weinstein [21] of a Le Brun-Poisson manifold over a contact manifold, and that of a Heisenberg-Poisson manifold over a symplectic (or Poisson) manifold. We show that the total space of a Jacobi bundle has a natural homogeneous Poisson structure, and that with each section of that bundle is associated a Hamiltonian vector field, defined on the total space of the bundle, tangent to the zero section, which projects onto the base manifold.

1. Introduction

The construction which associates a vector field (which is said to be Hamiltonian) with every smooth function on a symplectic manifold, is of central importance in symplectic geometry, and has many applications in mechanics and mathematical physics. It is now well known that such a construction exists also for manifolds with various structures, which need not be symplectic: odd-dimensional manifolds equipped with a contact 1-form, Poisson manifolds, Jacobi manifolds. Symplectic and Poisson manifolds are particular classes of Jacobi manifolds, but other important classes of manifolds, such as contact manifolds with no specified, globally defined contact form, are not. In section 3, we introduce the notion of a Jacobi bundle, which encompasses contact structures, as well as locally conformally symplectic structures. The notion of a Jacobi bundle is essentially equivalent to that of a conformal Jacobi structure on a manifold: the base space of a Jacobi bundle is a conformal Jacobi manifold, and conversely, under some mild assumptions, one can canonically build a Jacobi bundle over a given conformal Jacobi manifold [2, section 1.4]. We shall see (4.5) that in the particular cases when the given conformal Jacobi manifold is a contact manifold or a symplectic manifold, we obtain the constructions, made by A. Weinstein [21], of a Le Brun-Poisson manifold and of a Heisenberg-Poisson manifold, respectively. We shall prove (4.6) that on a Jacobi bundle, a Hamiltonian vector field is associated with every smooth section. This vector field is defined on the whole total space of the bundle, although, for its construction, we use a function defined on the open dense subset complementary to the zero section. It is tangent to the zero section, and projects onto the base space. Up to our knowledge, these properties do not seem to have been observed before. Finally (4.8), we compare some properties of Jacobi bundles with those of Lie algebroids, in the sense of Pradines [13].

2. Definitions and elementary properties

2.1. Definition. A *Jacobi structure* on a manifold M is defined by choosing a bilinear map from $C^\infty(M,\mathbf{R})\times C^\infty(M,\mathbf{R})$ into $C^\infty(M,\mathbf{R})$, called the *Jacobi bracket*, and denoted by

$$(f,g)\mapsto \{f,g\}\,,$$

which satisfies the following properties:

(i) it is skew-symmetric,

$$\{g,f\}=-\{f,g\}\,;$$

(ii) it satisfies the Jacobi identity,

$$\bigl\{f,\{g,h\}\bigr\}+\bigl\{g,\{h,f\}\bigr\}+\bigl\{h,\{f,g\}\bigr\}=0\,;$$

(iii) it is local, *i.e.*, the support of $\{f,g\}$ is contained in the intersection of the supports of f and of g.

A manifold with such a structure is called a *Jacobi manifold.*

2.2. Comments and properties. Jacobi manifolds were introduced independently by Kirillov [6] and Lichnerowicz [11], who used different, but equivalent, definitions.

Definition 2.1 of Jacobi manifolds is that introduced by Kirillov, who called them "local Lie algebras". Kirillov proved that on a Jacobi manifold M, the Jacobi bracket is expressed by a bidifferential operator of order at most one in each of its arguments. With the skew-symmetry property, this shows that there exist on M a vector field E and a skew-symmetric, contravariant 2-tensor Λ, both uniquely defined, such that for all f and $g\in C^\infty(M,\mathbf{R})$,

$$\{f,g\}=\Lambda(df,dg)+\langle f\,dg-g\,df,E\rangle\,. \tag{1}$$

Of course, Λ and E must satisfy some properties (indicated below) so that the Jacobi bracket satisfy the Jacobi identity.

A. Lichnerowicz [11] considered a manifold M equipped with a vector field E and a contravariant skew-symmetric 2-tensor Λ. He defined the Jacobi bracket of two functions f and $g\in C^\infty(M,\mathbf{R})$ by formula (1) above, and proved that the Jacobi bracket satisfies the Jacobi identity if and only if Λ and E satisfy the following two identities:

$$[E,\Lambda]=\mathcal{L}(E)\Lambda=0\,,\qquad [\Lambda,\Lambda]=2E\wedge\Lambda\,. \tag{2}$$

The bracket which appears in the above expressions is the Schouten bracket [15] [8]. Lichnerowicz defined a Jacobi manifold as a manifold M, equipped with a vector field E and a contravariant, skew-symmetric 2-tensor Λ, which satisfy identities (2). Clearly, this definition is equivalent to 2.1.

In what follows, such a Jacobi manifold will be denoted by (M, Λ, E).

2.3. Examples.

1. Let E be a smooth vector field on a manifold M. We set, for all f and $g \in C^\infty(M, \mathbf{R})$,

$$\{f, g\} = \langle f\,dg - g\,df, E\rangle\,.$$

One may easily verify that this bracket satisfies the Jacobi identity, and therefore defines a Jacobi structure on M.

2. Let (M, Ω) be a symplectic manifold. The bundle map

$$\Omega^\flat : TM \to T^*M\,, \qquad \Omega^\flat(X) = -i(X)\Omega\,, \tag{3}$$

is an isomorphism, since Ω is nondegenerate. Let $\Lambda^\sharp = (\Omega^\flat)^{-1} : T^*M \to TM$ be its inverse. The well known Poisson bracket of two functions f and $g \in C^\infty(M, \mathbf{R})$ is defined by

$$\{f, g\} = \Omega\big(\Lambda^\sharp(df), \Lambda^\sharp(dg)\big) = \big\langle dg, \Lambda^\sharp(df)\big\rangle = -\big\langle df, \Lambda^\sharp(dg)\big\rangle\,.$$

As it satisfies the Jacobi identity, we see that symplectic manifolds are a special class of Jacobi manifolds.

3. A Jacobi manifold (M, Λ, E) in which the vector field E vanishes identically is called a Poisson manifold [10] [20], and denoted by (M, Λ). A Jacobi manifold M is a Poisson manifold if and only if the Poisson bracket is a derivation (for the ordinary product of functions) in each of its arguments, *i.e.*, if and only if, for all $f, f_1, f_2, g, g_1, g_2 \in C^\infty(M, \mathbf{R})$,

$$\{f_1 f_2, g\} = f_1\{f_2, g\} + \{f_1, g\}f_2\,; \quad \{f, g_1 g_2\} = \{f, g_1\}g_2 + g_1\{f, g_2\}\,.$$

In particular, a symplectic manifold (M, Ω) (example 2) is a Poisson manifold: its 2-tensor Λ is such that the associated bundle map $\Lambda^\sharp : T^*M \to TM$, defined by $\langle \eta, \Lambda^\sharp \xi\rangle = \Lambda(\xi, \eta)$, is the inverse of the bundle map $\Omega^\flat : TM \to T^*M$ defined by equation (3).

4. A locally conformally symplectic structure [4] [9] on an even-dimensional manifold M is defined by a pair (Ω, ω), where Ω is a 2-form and ω a 1-form, such that Ω is everywhere of rank $2n = \dim M$, which satisfy

$$d\omega = 0\,, \qquad d\Omega + \omega \wedge \Omega = 0\,.$$

Let E be the unique vector field and Λ be the unique 2-tensor such that

$$i(E)\Omega = -\omega\,, \qquad i(\Lambda^\sharp \beta) = -\beta \quad \text{for every} \quad \beta \in T^*M\,.$$

Then (M, Λ, E) is a Jacobi manifold: this can be easily verified by observing that on a neighborhood of each point, there exists a function f such that $\omega = df$, and that the locally defined 2-form $\exp(f)\,\Omega$ is symplectic. See also example 6 below.

5. Let M be a $(2n+1)$-dimensional manifold equipped with a contact 1-form ω, *i.e.*, a Pfaffian form such that $\omega \wedge (d\omega)^n$ nowhere vanishes. Let E be the Reeb vector field [14], *i.e.*, the unique vector field on M such that

$$i(E)\omega = 1\,, \qquad i(E)d\omega = 0\,.$$

Let $\Lambda^\sharp : T^*M \to TM$ be the vector bundle map such that, for each $\xi \in T^*M$,

$$i(\Lambda^\sharp \xi)\omega = 0 \quad \text{and} \quad i(\Lambda^\sharp \xi)d\omega = -\bigl(\xi - \langle \xi, E\rangle \omega\bigr)\,.$$

We define the 2-tensor Λ by

$$\Lambda(\xi, \eta) = \langle \eta, \Lambda^\sharp \xi\rangle = -\langle \xi, \Lambda^\sharp \eta\rangle\,,$$

where ξ and η are two elements in T^*M which belong to the same fiber. Then Λ and E define a Jacobi structure on M, determined by the contact form ω.

6. Let (M, Λ, E) be a Jacobi manifold, and $a \in C^\infty(M, \mathbf{R})$ be a function which vanishes nowhere on M. For each pair (f, g) of functions in $C^\infty(M, \mathbf{R})$, we may set

$$\{f, g\}_a = \frac{1}{a}\{af, ag\}\,.$$

It can be seen easily that this new bracket is skew-symmetric, local, and satisfies the Jacobi identity. Therefore it defines another Jacobi structure on M, which is said to be *a-conformal* to the initially given one. The vector field E_a and the 2-tensor Λ_a which come with this new structure are given by

$$\Lambda_a = a\Lambda\,; \qquad E_a = aE + \Lambda^\sharp(da) = \Phi(a)\,.$$

The map Φ, which associates with any function f a vector field $\Phi(f)$, will be formally defined in 2.6.

When the initially given Jacobi structure on M comes from a symplectic 2-form Ω (example 2), the a-conformal Jacobi structure is such that

$$\Lambda_a^\sharp = a(\Omega^\flat)^{-1}\,; \qquad E_a = (\Omega^\flat)^{-1}(da) = \Lambda^\sharp(da)\,.$$

It is associated with the locally conformally symplectic structure (example 4) defined on M by the pair

$$\Omega_a = \frac{1}{a}\Omega\,, \qquad \omega_a = \frac{da}{a}\,.$$

Since ω_a is exact, such a structure on M is said to be *conformally symplectic.*

When the initially given Jacobi structure on M comes from a contact 1-form ω (example 5), the a-conformal Jacobi structure comes from the contact 1-form

$$\omega_a = \frac{1}{a}\omega \,.$$

2.4. Definition. Let (M_1, Λ_1, E_1) and (M_2, Λ_2, E_2) be two Jacobi manifolds, and $\varphi : M_1 \to M_2$ be a smooth map.

1. The map φ is said to be a *Jacobi map* if, for every pair (f, g) of functions on M_2,

$$\{f \circ \varphi, g \circ \varphi\}_1 = \{f, g\}_2 \circ \varphi \,.$$

2. The map φ is said to be a *conformal Jacobi map* if there exists a nowhere vanishing function a on M_1, such that φ is a Jacobi map when M_1 is equipped with the Jacobi structure a-conformal to the initially given one, *i.e.*, if, for every pair (f, g) of functions on M_2,

$$\{af \circ \varphi, ag \circ \varphi\}_1 = a\bigl(\{f, g\}_2 \circ \varphi\bigr) \,.$$

A map $\varphi : M_1 \to M_2$ is a Jacobi map if and only if the pairs (Λ_1, Λ_2) and (E_1, E_2) are compatible with φ.

2.5. Example. (I am indebted to A. Weinstein for pointing out this example, although it appears under a hidden form in [2].) Let (M, Ω) be a prequantizable symplectic manifold [16], [7], *i.e.*, a manifold with a symplectic 2-form whose cohomology class is integer. Let (Q, ω, π, M) be a prequantization of (M, Ω), *i.e.*, $\pi : Q \to M$ is a **T**-principal bundle with a connection form ω whose curvature is Ω. The connection form ω may be considered as a contact 1-form on Q. Then (Q, ω) and (M, Ω) are both particular Jacobi manifolds, and $\pi : Q \to M$ is a Jacobi map.

On Jacobi manifolds, just like on symplectic manifolds, we can associate a vector field (which is said to be *Hamiltonian*) with every smooth function.

2.6. Definition. Let (M, Λ, E) be a Jacobi manifold. We denote by $\Lambda^\sharp : T^*M \to TM$ the vector bundle map associated with Λ, *i.e.*, such that, for any $x \in M$, ξ and $\eta \in T^*_x M$,

$$\Lambda(\xi, \eta) = \bigl\langle \eta, \Lambda^\sharp(\xi) \bigr\rangle = -\bigl\langle \xi, \Lambda^\sharp(\eta) \bigr\rangle \,.$$

1. For any smooth function $f \in C^\infty(M, \mathbf{R})$, the vector field

$$\Phi(f) = \Lambda^\sharp(df) + fE$$

is called the *Hamiltonian vector field* associated with f, and the function f is called a *Hamiltonian* for the vector field $\Phi(f)$.

2. The *characteristic distribution* of (M, Λ, E) is the subset C of TM generated by the values of all the Hamiltonian vector fields.

2.7. Remarks.

1. The constant function equal to unity is a Hamiltonian for the vector field $E = \Phi(1)$.

2. By using 2.6.1, we see that the fiber $C_x = C \cap T_x M$ of the characteristic distribution C over a point $x \in M$ is the vector subspace of $T_x M$ generated by the vector $E(x)$ and the image of the linear map $\Lambda_x^\sharp : T_x^* M \to T_x M$.

3. The Jacobi manifold (M, Λ, E) is said to be *transitive* if its characteristic distribution is the whole of its tangent bundle TM. Lichnerowicz [11, 4] and Kirillov [6] have shown that transitive Jacobi manifolds are, according to the parity of their dimension, either locally conformally symplectic manifolds (example 2.3.5), or manifolds equipped with a contact 1-form (example 2.3.6).

2.8. Proposition. *Let (M, Λ, E) be a Jacobi manifold, and Φ be the map which associates with every function the corresponding Hamiltonian vector field.*

1. *The map Φ is a Lie algebra homomorphism, i.e.,*

$$\Phi\bigl(\{f, g\}\bigr) = \bigl[\Phi(f), \Phi(g)\bigr], \qquad f \text{ and } g \in C^\infty(M, \mathbf{R}).$$

2. *For any $f \in C^\infty(M, \mathbf{R})$, the first order differential operator $\Phi(f) - E.f$ is a derivation of the Jacobi Lie algebra $C^\infty(M, \mathbf{R})$ (equipped with the Jacobi bracket).In other words, for any f and $g \in C^\infty(M, \mathbf{R})$,*

$$\Phi(f).\{g, h\} - (E.f)\{g, h\} = \bigl\{\Phi(f).g - (E.f)g, h\bigr\} + \bigl\{g, \Phi(f).h - (E.f)h\bigr\}.$$

2.9. Theorem (Kirillov [6]). *The characteristic distribution of a Jacobi manifold (M, Λ, E) is completely integrable in the sense of Stefan* [17] *and Sussmann* [19]*, thus defines on M a Stefan foliation (i.e., a foliation whose leaves may not be all of the same dimension), called the characteristic foliation. Each leaf of this foliation has a unique transitive Jacobi structure such that its canonical injection into M is a Jacobi map (2.4).*

2.10. Remark. The last theorem, together with remark 2.7.3, shows that each leaf of the characteristic foliation of a Jacobi manifold is

- a locally conformally symplectic manifolds if its dimension is even,
- a manifold equipped with a contact 1-form if its dimension is odd.

For Poisson manifolds, the leaves of the characteristic foliation, all of even dimension, are symplectic manifolds.

3. Jacobi bundles

The following observations show that the notion of a Jacobi manifold may not be quite satisfactory for all practical purposes, and may need to be slightly generalized.

A contact structure on a $(2n+1)$-dimensional manifold M is defined by a vector subbundle P^* of rank 1 of its cotangent bundle T^*M such that any local, nowhere vanishing section ω of P^* is a contact 1-form (*i.e.*, $\omega \wedge (d\omega)^n \neq 0$ everywhere). In several important examples, such as the manifold of contact elements on $\mathbf{R}^{n+1}$, the contact structure of M cannot be defined by a single contact 1-form defined on the whole of M: one has to choose an open covering (U_i), $i \in I$, of M, such that on each U_i there exists a nowhere vanishing section ω_i of P^*. Of course,

$$\omega_j = f_{ji}\omega_i \qquad \text{on } U_i \cap U_j ,$$

where f_{ji} is a real valued, nowhere vanishing function on $U_i \cap U_j$. Clearly, Jacobi manifolds are not generalizations of contact manifolds: as shown by example 2.3.5, they are only generalizations of manifolds equipped with a specified, globally defined contact 1-form.

On a Jacobi manifold (M, Λ, E), the flow of a Hamiltonian vector field $\Phi(f)$, with $f \in C^\infty(M, \mathbf{R})$, is not in general a one-parameter local group of Jacobi automorphisms: it is only a one-parameter local group of *conformal* Jacobi automorphisms. This follows from the fact that $\Phi(f) - E.f$, and not $\Phi(f)$ itself, is a derivation of the Jacobi Lie algebra $C^\infty(M, \mathbf{R})$ (proposition 2.8). Therefore, when looking at Hamiltonian vector fields, we are led to use not only the Jacobi structure initially defined by Λ and E on M, but also all the Jacobi structures which are conformal to that structure, in the sense of example 2.3.6.

The appropriate generalization of the notion of a Jacobi manifold, already introduced by Kirillov [6], is the following.

3.1. Definition. Let (P, π, M) be a line bundle over a manifold M, *i.e.*, a vector bundle whose fibers $\pi^{-1}(x)$, $x \in M$, are one-dimensional. Let $\Gamma^\infty(\pi)$ be the vector space of global smooth sections of π. A Jacobi bundle structure on (P, π, M) is defined by choosing a bilinear map from $\Gamma^\infty(\pi) \times \Gamma^\infty(\pi)$ into $\Gamma^\infty(\pi)$, called the *Jacobi bracket*, denoted by

$$(s_1, s_2) \mapsto \{s_1, s_2\} ,$$

which satisfies the following properties:

(i) it is skew-symmetric,

$$\{s_2, s_1\} = -\{s_1, s_2\} ;$$

(ii) it satisfies the Jacobi identity,

$$\{s_1, \{s_2, s_3\}\} + \{s_2, \{s_3, s_1\}\} + \{s_3, \{s_1, s_2\}\} = 0 ;$$

(iii) it is local, *i.e.*, the support of $\{s_1, s_2\}$ is contained in the intersection of the supports of s_1 and s_2.

When equipped with such a structure, (P, π, M) is called a *Jacobi bundle*.

3.2. Definition. Let (P_1, π_1, M_1) and (P_2, π_2, M_2) be two Jacobi bundles, and $\varphi : P_1 \to P_2$ be a vector bundle map, *i.e.*, a smooth map whose restriction to each fiber of P_1 is a linear isomorphism of that fiber onto a fiber of P_2. Let $\widetilde{\varphi} : M_1 \to M_2$ be the corresponding base map, *i.e.*, the unique smooth map such that $\pi_2 \circ \varphi = \widetilde{\varphi} \circ \pi_1$. Then φ is said to be a *Jacobi bundle map* if, for every pair (s_2, s_2') of sections of π_2,

$$\{s_2, s_2'\} \circ \widetilde{\varphi} = \varphi \circ \{s_1, s_1'\}\,,$$

where s_1 and s_1' are the sections of π_1 such that $s_2 \circ \widetilde{\varphi} = \varphi \circ s_1$, $s_2' \circ \widetilde{\varphi} = \varphi \circ s_1'$.

3.3. Remarks.

1. Let (M, Λ, E) be a Jacobi manifold, $P = \mathbf{R} \times M$, and $\pi : P \to M$ be the second projection. Every function $f \in C^\infty(M, \mathbf{R})$ may be identified with a section $x \mapsto \bigl(f(x), x\bigr)$ of π. Clearly, (P, π, M) is a (trivial) Jacobi bundle.

Let (M_i, Λ_i, E_i) ($i = 1$ or 2) be two Jacobi manifolds, $(P_i = \mathbf{R} \times M_i, \pi_i, M_i)$ be the corresponding trivial Jacobi bundles defined as above. Let $\widetilde{\varphi} : M_1 \to M_2$ be a Jacobi map. Then the map $\varphi : P_1 \to P_2$ defined by

$$\varphi(z, x) = \bigl(z, \widetilde{\varphi}(x)\bigr)$$

is a Jacobi bundle map.

2. Conversely, let (P, π, M) be a Jacobi bundle, and U an open subset of M on which there exists a nowhere vanishing section $s_0 : U \to P$ of π. Every function $f \in C^\infty(U, \mathbf{R})$ may be associated with the section fs_0 of π. For every pair (f, g) of functions on U, let $\{f, g\}$ be the unique function on U such that

$$\{fs_0, gs_0\} = \{f, g\}s_0\,.$$

Clearly this defines a Jacobi structure on U.

Let $s_0' : U \to P$ be another nowhere vanishing section of π defined on U, and let $a : U \to \mathbf{R}$ be the unique, nowhere vanishing function on U such that for each $x \in U$,

$$s_0'(x) = a(x)s(x)\,.$$

Then the Jacobi structure on U defined by choosing s_0' as reference-section, instead of s_0, is a-conformal to the one defined by choosing s_0 as a reference-section.

Let φ be a Jacobi bundle map from a Jacobi bundle (P_1, π_1, M_1) into another Jacobi bundle (P_2, π_2, M_2). Let $\widetilde{\varphi} : M_1 \to M_2$ be the associated base map. Let U_i be an open subset of M_i on which there exists a nowhere vanishing section $s_{i0} : U_i \to P_i$ of π_i ($i = 1$ or 2), such that $\widetilde{\varphi}(U_1) \subset U_2$. We equip each U_i ($i = 1$ or 2) with the Jacobi structure associated with s_{i0}, as indicated above. Let $a : U_1 \to \mathbf{R}$ be the unique, nowhere vanishing function on U_1 such that for each $x \in U_1$

$$s_{20} \circ \widetilde{\varphi}(x) = a(x)\varphi \circ s_{10}(x) = \varphi \circ \bigl(as_{10}\bigr)(x)\,.$$

Then $\widetilde{\varphi}$ is an a-conformal Jacobi map from U_1 into U_2.

3. The notion of a Jacobi bundle is essentially equivalent to that of a conformal Jacobi structure, introduced for example in [2].

3.4. Examples.

1. Let M be a $(2n+1)$-dimensional manifold, equipped with a contact structure defined by a vector sub-bundle P^* of rank 1 of T^*M. Let $(P^*)^0$ be the annihilator of P^*, $P = TM/(P^*)^0$ be the dual bundle of P^*, and $\pi : P \to M$ be the canonical projection. There exists on P a natural Jacobi bundle structure, defined as follows. Let s_1 and s_2 be two (global) sections of π. Let U be an open subset of M on which there exists a nowhere vanishing section $\omega_0 : U \to P^*$ of P^*, and let $s_0 : U \to P$ be the dual section of P, *i.e.*, the nowhere vanishing section such that, for each $x \in U$, $\langle \omega_0(x), s_0(x) \rangle = 1$. There exist two functions f_1 and f_2 on U, both uniquely defined, such that

$$s_1 = f_1 s_0 \,, \qquad s_2 = f_2 s_0 \,.$$

We define $\{s_1, s_2\}$ on U by setting

$$\{s_1, s_2\}\big|_U = \{f_1, f_2\}_{\omega_0} s_0 \,,$$

where $\{f_1, f_2\}_{\omega_0}$ is the Jacobi bracket of the two functions f_1 and f_2, for the Jacobi structure on U associated with the contact 1-form ω_0.

The bracket $\{s_1, s_2\}\big|_U$ does not depend on the choice of ω_0: if we replace ω_0 by $\omega'_0 = a^{-1}\omega_0$, where a is a nowhere vanishing function on U, s_0, f_1 and f_2 are replaced by $s'_0 = a s_0$, $f'_1 = a^{-1} f_1$ and $f'_2 = a^{-1} f_2$, respectively. But as seen in example 2.3.6,

$$\{f'_1, f'_2\}_{\omega'_0} = \frac{1}{a}\,\{af'_1, af'_2\}_{\omega_0} = \frac{1}{a}\,\{f_1, f_2\}_{\omega_0} \,,$$

thus

$$\{f'_1, f'_2\}_{\omega'_0} s'0 = a\{f'_1, f'_2\}_{\omega'_0} s_0 = \{f_1, f_2\}_{\omega_0} s_0 \,.$$

Therefore, by using this construction for each open subset of an open covering (U_i), $i \in I$, of M, such that on each U_i there exists a nowhere vanishing section of P^*, we obtain a globally defined Jacobi bundle structure on (P, π, M).

2. Let M be a $2n$-dimensional manifold, and let P^* be a vector sub-bundle of rank 1 of $\bigwedge^2 T^*M$ (the bundle of exterior 2-forms on M), such that any nowhere vanishing local section $\Omega : U \to P^*$ of that bundle is everywhere of rank $2n$ and such that there exists, on the open subset U of M, a 1-form ω which satisfy

$$d\Omega + \omega \wedge \Omega = 0 \,; \qquad d\omega = 0 \,.$$

We observe that if $\Omega : U \to P^*$ satisfies these conditions, then for any nowhere vanishing function a on U, the section $a\Omega$ satisfies the same conditions, since

$$d(a\Omega) + (\omega - a^{-1}da) \wedge (a\Omega) = 0 \,, \qquad d(\omega - a^{-1}da) = 0 \,.$$

We observe also that if $n > 1$, the 1-form ω is completely determined once Ω is chosen. For $n = 1$, we will impose $\omega = 0$, and the conditions which must be satisfied by P^* are trivially satisfied.

The structure so defined is obviously a slight generalization of that of a locally conformally symplectic structure (example 2.3.4).

Let P be the dual bundle of P^*, and $\pi : P \to M$ be the canonical projection. Then (P, π, M) has a natural Jacobi bundle structure, which may be defined in a way similar to that in the previous example: we take two sections s_1 and s_2 of π; for any local nowhere vanishing section $\Omega : U \to P^*$, we take the dual section $\sigma : U \to P$ such that $\langle \Omega, \sigma \rangle = 1$; we take the functions f_1 and f_2 such that $s_1 = f_1\sigma$ and $s_2 = f_2\sigma$; then we define $\{s_1, s_2\}$ on U as being the section $\{f_1, f_2\}\sigma$, where $\{f_1, f_2\}$ is the Jacobi bracket relative to the locally conformally symplectic structure defined on U by Ω.

Such a structure will be called a *locally conformally symplectic bundle structure* over M. Clearly, a locally conformally symplectic structure on M is a trivial locally conformally symplectic bundle structure over M, with a specified nowhere vanishing section of that trivial bundle.

3.5. Remarks.

1. There are well known examples of contact manifolds with no globally defined contact 1-form corresponding to their contact structure. In a similar way, there exist locally conformally symplectic bundle structures which are not trivial: for example, the dual bundle of the area-element bundle on a non-orientable 2-dimensional manifold.

2. Let M be a $(2n + 1)$-dimensional manifold with a contact structure defined by the rank 1 sub-bundle P^* of T^*M. Stong [18] has shown that if n is even, then M is orientable, and that if n is odd, then M is orientable if and only if the bundle P^* is trivial. Similarly, let (P, π, M) be a locally conformally symplectic bundle over a $(2n)$-dimensional manifold M. One can prove (F. Guédira and A. Lichnerowicz [4]) that if n is even, then M is orientable, and that if n is odd, then M is orientable if and only if the bundle P is trivial.

3. Let us give an example of an orientable $2n$-dimensional manifold, wit $n = 2p$ even, over which there exists a nontrivial locally conformally symplectic bundle structure. We define on $\mathbf{R}^4$ an equivalence relation, by saying that (x_1, x_2, x_3, x_4) and (x'_1, x'_2, x'_3, x'_4) are equivalent if and only if

$$x'_1 - x_1 \in \mathbf{Z}\,, \quad x'_2 - \epsilon x_2 = 0\,, \quad x'_3 - x_3 = 0\,, \quad x'_4 - \epsilon x_4 = 0\,,$$

with

$$\epsilon = \begin{cases} 1 & \text{if } x'_1 - x_1 \text{ is even,} \\ -1 & \text{if } x'_1 - x_1 \text{ is odd.} \end{cases}$$

The quotient of $\mathbf{R}^4$ by this equivalence relation is a manifold M, which may be identified with the cotangent bundle to a Möbius stripe; its canonical symplectic 2-form is the projection of the 2-form $-(dx_1 \wedge dx_3 + dx_2 \wedge dx_4)$ on $\mathbf{R}^4$. But if, instead of that 2-form, we consider the 2-form $dx_1 \wedge dx_2 + dx_3 \wedge dx_4$ on $\mathbf{R}^4$, we see that by projection it defines a nontrivial locally conformally symplectic bundle structure on M.

4. Jacobi bundles and homogeneous Poisson manifolds

The close relations between Jacobi manifolds and homogeneous Poisson manifolds were already indicated in [2]. In fact, it appears that even closer relations exist between Jacobi bundles and homogeneous Poisson manifolds. We use here the term "homogeneous Poisson manifold" to agree with previous usage, though this name may be misleading. Let us recall its definition

4.1. Definitions.

1. A *homogeneous Poisson manifold* (P, Λ, Z) is a Poisson manifold (P, Λ) with a vector field Z on P such that

$$\mathcal{L}(Z)\Lambda = -\Lambda .$$

2. Let (P_1, Λ_1, Z_1) and (P_2, Λ_2, Z_2) be two homogeneous Poisson manifolds. A *strict homogeneous Poisson map* is a Poisson map $\varphi : P_1 \to P_2$ such that, for every $x \in P_1$, $T_x\varphi\bigl(Z_1(x)\bigr) = Z_2\bigl(\varphi(x)\bigr)$.

4.2. Example. Let $\mathcal{G}$ be a real, finite dimensional Lie algebra, and $(\mathcal{G}^*, \Lambda)$ be its dual space equipped with its canonical Lie-Poisson structure. We recall that the Poisson bracket of two functions f and $g \in C^\infty(\mathcal{G}^*, \mathbf{R})$ is given by

$$\{f, g\}(x) = \Lambda(x)\bigl(df(x), dg(x)\bigr) = \bigl\langle x, \bigl[df(x), dg(x)\bigr]\bigr\rangle , \qquad x \in \mathcal{G}^* ,$$

where the differentials $df(x)$ and $dg(x)$ are considered as elements in $\mathcal{G}$.

Let Z be the Liouville vector field on $\mathcal{G}^*$, *i.e.*, the vector field given by $Z(x) = x$ for each $x \in \mathcal{G}^*$. Then $(\mathcal{G}^*, \Lambda, Z)$ is a homogeneous Poisson manifold.

A. Lichnerowicz [11] and A.-M. Justino [5] have shown that with any Jacobi manifold (M, Λ, E) one can associate a homogeneous Poisson manifold (P, Λ_P, Z) by setting

$$P = \mathbf{R} \times M , \quad Z(t, x) = \frac{\partial}{\partial t} , \quad \Lambda_P(t, x) = \exp(-t)\bigl(\Lambda(x) + Z(t, x) \wedge E(x)\bigr) ,$$

where t is the usual coordinate on $\mathbf{R}$, and $x \in M$.

This construction is not quite satisfactory, because using a product of M with $\mathbf{R}$ may seem un-natural, and because it does not allow to lift as Poisson maps all the conformal Jacobi maps, but only those whose conformal factor is strictly positive.

The following construction allows us to associate with any Jacobi bundle a homogeneous Poisson manifold, in a way which is more natural and fully functorial.

4.3. Proposition. *Let (P, π, M) be a Jacobi bundle. There exists on P a unique Poisson structure (P, Λ), with the following property. Let s_1 and s_2 be two sections of π,*

F_1 and F_2 be the functions, defined on the open subset P_0 of P complementary to the zero section, such that

$$s_i\bigl(\pi(x)\bigr) = F_i(x)\,x\,, \quad i = 1 \text{ or } 2\,, \quad x \in P_0\,.$$

Then the bracket $\{F_1, F_2\}$ is such that

$$\{s_1, s_2\}\bigl(\pi(x)\bigr) = \{F_1, F_2\}(x)\,x\,, \qquad x \in P_0\,.$$

Moreover, this Poisson structure on P is homogeneous for the vector field Z on P, opposite to the Liouville vector field (i.e., for each $x \in P$, $Z(x)$ is the vector tangent to the fiber containing x and equal to $-x$).

Proof. Let U be an open subset of M on which there exists a nowhere vanishing section $s_0 : U \to P_0$ of π. Let f_1 and f_2 be the real valued functions on U such that

$$s_1 = f_1 s_0\,, \qquad s_2 = f_2 s_0\,.$$

By choosing s_0, we have defined a Jacobi structure on U such that

$$\{f_1, f_2\} s_0 = \{f_1 s_0, f_2 s_0\}\,.$$

We will denote by Λ_{s_0} and E_{s_0} the corresponding tensor and vector field.

The map from $\mathbf{R} \times U$ onto $\pi^{-1}(U)$,

$$(\lambda, y) \mapsto \lambda s_0(y)$$

is a trivialization of P over U. The functions F_1 and F_2, defined on $P_0 \cap \pi^{-1}(U)$, may thus be considered as functions on $(\mathbf{R}\backslash\{0\}) \times U$. They are given by

$$F_i(\lambda, y) = \lambda^{-1} f_i(y)\,, \qquad i = 1 \text{ or } 2\,.$$

We want to define a Poisson tensor Λ_P on P, such that

$$\begin{aligned}
\Lambda_P(\lambda, y)\bigl(dF_1(\lambda, y), dF_2(\lambda, y)\bigr) &= \{F_1, F_2\}(\lambda, y)\\
&= \lambda^{-1}\{f_1, f_2\}(y)\\
&= \lambda^{-1}\Bigl(\Lambda_{s_0}(y)\bigl(df_1(y), df_2(y)\bigr)\\
&\qquad + \bigl\langle f_1(y) df_2(y) - f_2(y) df_1(y), E_{s_0}(y)\bigr\rangle\Bigr)\,.
\end{aligned}$$

Since $dF_i(\lambda, y) = \lambda^{-1}\, df_i(y) - \lambda^{-2} f_i(y)\, d\lambda$, and since (f_1, f_2) can be any pair of functions on U, the only possible choice for Λ_P is

$$\Lambda_P(\lambda, y) = \lambda \Lambda_{s_0}(y) - \lambda^2 \frac{\partial}{\partial \lambda} \wedge E_{s_0}(y)\,. \tag{4}$$

This defines indeed a Poisson tensor on $\pi^{-1}(U)$, because the change of variables $\lambda = e^{-t}$ yields

$$\Lambda_P(t, y) = e^{-t}\left(\Lambda_{s_0}(y) + \frac{\partial}{\partial t} \wedge E_{s_0}(y)\right)\,,$$

which is the formula used by Lichnerowicz [11] and Justino [5], with $Z = \frac{\partial}{\partial t} = -\lambda \frac{\partial}{\partial \lambda}$.

The above calculations are local, and show that Λ_P is a Poisson tensor. But the way in which the Poisson structure on P was defined is global, so there is no need to verify what happens when we replace the open subset U and the nowhere vanishing section s_0 by other data U' and s'_0: we know that under such a change, Λ_P remains invariant.

Moreover, the formula obtained above for $\Lambda_P(\lambda, y)$ shows that the Poisson structure, initially defined on P_0, extends smoothly to the zero section, where Λ_P vanishes. Therefore Λ_P defines a Poisson structure on the whole of P, and since $Z = -\lambda \frac{\partial}{\partial \lambda}$, this Poisson structure is homogeneous with respect to Z. □

4.4. Proposition. *Let (P_1, π_1, M_1) and (P_2, π_2, M_2) be two Jacobi bundles. A smooth map $\varphi : P_1 \to P_2$ is a Jacobi bundle map if and only if it is a strict homogeneous Poisson map (P_1 and P_2 being endowed with their homogeneous Poisson structures defined in 4.3) which maps every nonzero element in P_1 onto a nonzero element in P_2.*

Proof. Let $(P_1, \Lambda_{P_1}, Z_{P_1})$ and $(P_2, \Lambda_{P_2}, Z_{P_2})$ be the homogeneous Poisson structures on P_1 and P_2 defined in 4.3.

1. If φ is a Jacobi bundle map, its restriction to any fiber of π_1 is a linear isomorphism of that fiber onto a fiber of π_2. Therefore φ maps every nonzero element in P_1 onto a nonzero element in P_2 and satisfies

$$\varphi_* Z_1 = Z_2 .$$

By using local trivializations of (P_i, π_i, M_i), $i = 1$ or 2, and taking into account remark 3.2 and the local expressions of Λ_{P_i} obtained in the proof of Proposition 4.3, we obtain easily

$$\varphi_* \Lambda_{P_1} = \Lambda_{P_2} .$$

2. Conversely, if we assume that φ is a strict homogeneous Poisson map, it satisfies

$$\varphi_* Z_1 = Z_2 ;$$

thus its restriction to each fiber of π_1 is a linear map of that fiber into a fiber of π_2. If in addition we assume that φ maps every nonzero element in P_1 onto a nonzero element in P_2, its restriction to each fiber of π_1 is a linear isomorphism onto the corresponding fiber of π_2. In other words, φ is a vector bundle map. Let $\widetilde{\varphi} : M_1 \to M_2$ be the corresponding base map.

Let s_2 and s'_2 be two sections of π_2, F_2 and F'_2 be the associated real valued functions on $P_{20} = P_2$ minus the zero section, such that for each $y \in P_{20}$

$$s_2 \circ \pi_2(y) = F_2(y) y , \qquad s'_2 \circ \pi_2(y) = F'_2(y) y .$$

Let s_1 and s'_1 be the sections of π_1 such that $s_2 \circ \widetilde{\varphi} = \varphi \circ s_1$, $s'_2 \circ \widetilde{\varphi} = \varphi \circ s'_1$. We observe that the real valued functions on $P_{10} = P_1$ minus the zero section associated with s_1 and s'_1 are $F_2 \circ \varphi$ and $F'_2 \circ \varphi$, respectively. By proposition 4.3, for each $x \in P_1$, we have

$$\begin{aligned}\{s_2, s'_2\} \circ \widetilde{\varphi} \circ \pi_1(x) &= \{s_2, s'_2\} \circ \pi_2 \circ \varphi(x)\\ &= \big(\{F_2, F'_2\} \circ \varphi(x)\big)\varphi(x)\\ &= \varphi\Big(\{F_2 \circ \varphi, F'_2 \circ \varphi\}(x)\, x\Big)\\ &= \varphi \circ \{s_1, s'_1\} \circ \pi_1(x)\,. \quad \square\end{aligned}$$

4.5. Remark. The homogeneous Poisson structure on the total space P of a Jacobi bundle (P, π, M) was already defined by F. Guédira and A. Lichnerowicz [4], and was obtained independently in [12], where I defined it on P_0 only: I did not observe at that time that it extends smoothly to the zero section. I became aware of that property (which is quite clearly stated in the paper by F. Guédira and A. Lichnerowicz) by reading A. Weinstein's paper [21]. Weinstein uses the same construction in two particular cases: when M is a contact manifold, in which case he attributes that construction (already made in [12]) to Le Brun; when M is a symplectic or a Poisson manifold, in which case he obtains what he calls a Heisenberg-Poisson manifold.

The following proposition shows that on a Jacobi bundle, one may associate a vector field with every section.

4.6. Proposition. *Let (P, π, M) be a Jacobi bundle. With every section s of π, we can associate a smooth vector field X_s on P, characterized by the following property. Let (P, Λ_P, Z) be the homogeneous Poisson structure on P defined in 4.4, and $F : P_0 \to \mathbf{R}$ be the function, defined on $P_0 = P$ minus the zero section, such that*

$$s \circ \pi(x) = F(x)\, x \quad \text{for all } x \in P_0\,.$$

Then X_s restricted to P_0 is equal to $\Lambda_P^\sharp(dF)$. We will say that X_s is the Hamiltonian vector field on P associated with the section s.

Proof. The vector field $\Lambda_P^\sharp(dF)$ is defined on the open dense subset P_0 of P, so we need only prove that it extends smoothly to the zero section. Let $s_0 : U \to P$ be a nowhere vanishing section of π, defined on an open subset U of M. We identify $\mathbf{R} \times U$ with $\pi^{-1}(U)$, by means of the map $(\lambda, x) \mapsto \lambda s_0(x)$. Let $f : U \to \mathbf{R}$ be the unique function such that

$$s(y) = f(y) s_0(y) \quad \text{for all } y \in U\,.$$

On $\pi^{-1}(U)$, identified with $\mathbf{R} \times U$, the function F may be written as

$$F(\lambda, y) = \lambda^{-1} f(y)\,.$$

Using expression (4) of Λ_P, we obtain

$$\Lambda_P^\sharp\big(dF(\lambda, y)\big) = \Lambda_{s_0}^\sharp\big(df(y)\big) + f(y) E_{s_0}(y) - \big\langle df(y), E_{s_0}(y)\big\rangle Z(\lambda, y)\,,$$

which proves that $\Lambda_P^\sharp(dF)$ extends smoothly to the zero section. □

The following proposition states some properties of Hamiltonian vector fields.

4.7. Proposition. *The assumptions and notations are those of proposition 4.6.*

1. *The Hamiltonian vector field X_s associated with a section s of π is projectible by π on M, i.e., there exists a unique vector field $\widetilde{X}_s$ on M such that $\pi_*(X_s) = \widetilde{X}_s$.*

2. *The Hamiltonian vector field X_s is tangent to the zero section, and its restriction to that zero section (identified with M) is equal to its projection $\widetilde{X}_s$.*

3. *The map $s \mapsto X_s$ is a Lie algebra homomorphism.*

Proof. All these properties are easy consequences of the local expressions given in the proofs of 4.4 and 4.6. □

4.8. Remarks. The assumptions and notations are those of propositions 4.6 and 4.7.

1. Let U be an open subset of M on which the section s nowhere vanishes, and let (U, Λ_U, E_U) be the corresponding Jacobi manifold structure on U, defined in remark 3.3.2. Then the restriction of $\widetilde{X}_s$ to U is equal to E_U.

2. The Hamiltonian vector field X_s associated with any section s of π satisfy

$$[Z, X_s] = 0\,.$$

3. The map $s \mapsto \widetilde{X}_s$ is a Lie algebra homomorphism from the Lie algebra of sections of π into the Lie algebra of vector fields on M. This property follows from 4.7.3, since the projection $X_s \mapsto \widetilde{X}_s$ is a Lie algebra homomorphism. However, $s \mapsto \widetilde{X}_s$ does not come from a vector bundle map from P into TM, because if f is a function on M and s a section of π, $\widetilde{X}_{fs}$ is not in general equal to $f\widetilde{X}_s$. In other words, (P, π, M) is not a Lie algebroid in the sense of Pradines [13] [1].

4. Dazord and Sondaz define in [3] a Lie-Poisson structure as a vector bundle whose total space is endowed with a Poisson structure which is homogeneous with respect to the Liouville vector field. They show that the dual bundle of a Lie-Poisson structure is a Lie algebroid (and conversely). This result is not valid for a Jacobi bundle (P, π, M), because the Poisson structure on P is homogeneous with respect to the *opposite* of the Liouville vector field on P. If we consider the dual bundle P^* of P, and the map which associates with every nonzero element z of P the element α in the corresponding fiber of P^* such that $\langle \alpha, z\rangle = 1$, we obtain a Poisson structure on the open subset P_0^* of P^* complementary to the zero section, which is homogeneous with respect to the Liouville vector field. But that structure does not extend smoothly to the zero section. That explains why Jacobi bundles are not Lie algebroids.

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