Master symmetries and bi-Hamiltonian structures for the relativistic Toda lattice

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Abstract

We define a bi-Hamiltonian formulation for the relativistic Toda lattice with a recursion operator on $\mathbb{R}^{2n}$. We use a theorem of W. Oevel to generate higher order Poisson tensors and master symmetries for the relativistic Toda lattice. These Poisson tensors and master symmetries reduce to $\mathbb{R}^{2n-1}$.

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Introduction

The relativistic Toda lattice (RTL) was introduced by S. N. Ruijsenaars [1] and has been studied by many authors, in particular M. Bruschi and O. Ragnisco [2, 3], W. Oevel *et al.* [4], Yu. B. Suris [5] and P. Damianou [6]. It is a finite-dimensional completely integrable bi-Hamiltonian system. Its bi-Hamiltonian formulation and its complete integrability were proven by using various methods: Lax representation [3], [6], master symmetries [4], [6], recursion operators [2], [4].

Master symmetries were introduced by A. S. Fokas and B. Fuchssteiner [9] and were also studied by W. Oevel and B. Fuchssteiner [10] and B. Fuchssteiner [11].

In this paper we obtain a bi-Hamiltonian formulation for the RTL by introducing two compatible Poisson tensors on $\mathbb{R}^{2n}$ which, by a suitable projection map onto $\mathbb{R}^{2n-1}$, reduce to the two compatible Poisson tensors of the RTL. Since one of the Poisson structures introduced on $\mathbb{R}^{2n}$ is nondegenerate, we have a recursion operator and the bi-Hamiltonian structure of the RTL is in fact multi-Hamiltonian. Then, using a method introduced by R. L. Fernandes [7] for the non-relativistic Toda lattice, based on a theorem due to W. Oevel [8], we determine master symmetries for the RTL. Our results answer a question put by P. Damianou in [6].

In this paper, all the manifolds, maps, vector and tensor fields are assumed to be smooth. Let us recall that a *bi-Hamiltonian manifold* is a manifold M equipped with two compatible Poisson tensors Λ_0 and Λ_1; it is denoted by $(M, \Lambda_0, \Lambda_1)$. A vector field X on M is said to be a *bi-Hamiltonian vector field* if it is Hamiltonian with respect to both Poisson structures. A *recursion operator* for $(M, \Lambda_0, \Lambda_1)$ is a vector bundle map $R : TM \to TM$ such that $\Lambda_1^\sharp = R \circ \Lambda_0^\sharp$, where $\Lambda_0^\sharp : T^*M \to TM$ and $\Lambda_1^\sharp : T^*M \to TM$ are the vector bundle maps associated with the Poisson tensors Λ_0 and Λ_1. A bi-Hamiltonian manifold for which there exists a recursion operator is called a *Poisson-Nijenhuis manifold* [12].

1 A bi-Hamiltonian formulation for the relativistic Toda lattice

We consider $\mathbb{R}^{2n}$ with coordinates $(q^1, \ldots, q^n, p_1, \ldots, p_n)$ and the canonical Poisson tensor

$$\Lambda_1 = \sum_{i=1}^{n} \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i}. \tag{1}$$

Following Yu. B. Suris [5], we take $(c_1, \ldots, c_{n-1}, d_1, \ldots, d_n)$ as variables on $\mathbb{R}^{2n-1}$, with

$$c_i = \exp(q^i - q^{i+1} + p_i) \qquad \text{and} \qquad d_i = \exp(p_i) \tag{2}$$

and we denote by $\pi : \mathbb{R}^{2n} \to \mathbb{R}^{2n-1}$ the map

$$\pi : (q^1, \ldots, q^n, p_1, \ldots, p_n) \mapsto (c_1, \ldots, c_{n-1}, d_1, \ldots, d_n). \tag{3}$$

The relativistic Toda lattice is a finite-dimensional integrable system. The equations of motion of the RTL are

$$\left\{ \begin{array}{l} \dot{c}_i = c_i(d_{i+1} - d_i + c_{i+1} - c_{i-1}) \\ \dot{d}_i = d_i(c_i - c_{i-1}), \end{array} \right. \tag{4}$$

where $i = 1, \ldots, n$ and, by convention, $c_0 = c_n = 0$.

The RTL is a bi-Hamiltonian system with respect to the following compatible Poisson tensors on $\mathbb{R}^{2n-1}$,

$$\bar{\Lambda}_0 = \sum_{i=1}^{n-1} c_i \left(\frac{\partial}{\partial c_i} \wedge \left(\frac{\partial}{\partial d_i} - \frac{\partial}{\partial d_{i+1}} \right) + \frac{\partial}{\partial d_i} \wedge \frac{\partial}{\partial d_{i+1}} \right) \tag{5}$$

and

$$\bar{\Lambda}_1 = \sum_{i=1}^{n-1} c_i \frac{\partial}{\partial c_i} \wedge \left(-c_{i+1} \frac{\partial}{\partial c_{i+1}} + d_i \frac{\partial}{\partial d_i} - d_{i+1} \frac{\partial}{\partial d_{i+1}} \right), \tag{6}$$

and the bi-Hamiltonian vector field

$$\begin{aligned} \bar{\Lambda}_0^{\#}(d\bar{H}_1) &= \bar{\Lambda}_1^{\#}(d\bar{H}_0) \\ &= \sum_{i=1}^{n} \Big((c_i c_{i+1} - c_{i-1} c_i + c_i d_{i+1} - c_i d_i) \frac{\partial}{\partial c_i} \\ &\qquad + (c_i d_i - c_{i-1} d_i) \frac{\partial}{\partial d_i} \Big), \end{aligned} \tag{7}$$

with Hamiltonians

$$\bar{H}_0 = \sum_{i=1}^{n} (c_i + d_i) \quad \text{and} \quad \bar{H}_1 = \sum_{i=1}^{n} (c_{i-1}(c_i + d_i) + \frac{1}{2}(c_i + d_i)^2), \tag{8}$$

where, by convention, $c_0 = c_n = 0$ in (6), (7) and (8).

We remark that the Poisson bracket associated with $\bar{\Lambda}_0$ is linear:

$$\{c_i, d_i\}_0 = c_i; \quad \{c_i, d_{i+1}\}_0 = -c_i; \quad \{d_i, d_{i+1}\}_0 = c_i,$$

while the Poisson bracket corresponding to $\bar{\Lambda}_1$ is quadratic:

$$\{c_i, c_{i+1}\}_1 = -c_i c_{i+1}; \quad \{c_i, d_i\}_1 = c_i d_i; \quad \{c_i, d_{i+1}\}_1 = -c_i d_{i+1}.$$

A simple computation shows that the map

$$\pi : (\mathbb{R}^{2n}, \Lambda_1) \to (\mathbb{R}^{2n-1}, \bar{\Lambda}_1)$$

is a Poisson morphism, that is, the canonical Poisson tensor Λ_1 on $\mathbb{R}^{2n}$ reduces to the quadratic Poisson tensor $\bar{\Lambda}_1$ on $\mathbb{R}^{2n-1}$. This is not the case of the non-relativistic Toda lattice where the canonical Poisson bracket associated with Λ_1 reduces to a Poisson bracket on $\mathbb{R}^{2n-1}$, that is linear.

Our goal is to provide a bi-Hamiltonian formulation on $\mathbb{R}^{2n}$ for the RTL with a recursion operator.

The first step is to define a Poisson tensor on $\mathbb{R}^{2n}$, compatible with the canonical Poisson tensor Λ_1 and that reduces to the linear Poisson tensor $\bar{\Lambda}_0$ on $\mathbb{R}^{2n-1}$ given by (5).

Proposition 1.1 *Let Λ_0 be the following bivector on $\mathbb{R}^{2n}$:*

$$\begin{aligned}\Lambda_0 &= \sum_{i=1}^{n} \exp(-p_i) \frac{\partial}{\partial q^i} \wedge \left(\frac{\partial}{\partial p_i} + \sum_{j=i+1}^{n} \frac{\partial}{\partial q^j} \right) \\ &+ \sum_{i=1}^{n-1} \exp(q^i - q^{i+1} - p_{i+1}) \left(\left(\frac{\partial}{\partial p_i} + \frac{\partial}{\partial q^{i+1}} \right) \wedge \left(\frac{\partial}{\partial p_{i+1}} + \sum_{j=i+2}^{n} \frac{\partial}{\partial q^j} \right) \right. \\ &\qquad \left. - \frac{\partial}{\partial p_{i+1}} \wedge \sum_{j=i+2}^{n} \frac{\partial}{\partial q^j} \right) . \end{aligned} \tag{9}$$

Then,

(i) $[\Lambda_0, \Lambda_0] = 0$ *and* $[\Lambda_0, \Lambda_1] = 0$;

(ii) $\Lambda_1^\#(dH_0) = \Lambda_0^\#(dH_1)$, *with*

$$H_0 = \sum_{i=1}^{n} \left(\exp(q^i - q^{i+1} + p_i) + \exp(p_i) \right)$$

and

$$\begin{aligned} H_1 &= \sum_{i=1}^{n} \left(\frac{1}{2} \left(\exp(q^i - q^{i+1} + p_i) + \exp(p_i) \right)^2 \right. \\ &\qquad \left. + \exp(q^{i-1} - q^i + p_{i-1}) \left(\exp(q^i - q^{i+1} + p_i) + \exp(p_i) \right) \right) , \end{aligned}$$

where, by convention, $q^0 = -\infty$ and $q^{n+1} = +\infty$;

(iii) *the map $\pi : (\mathbb{R}^{2n}, \Lambda_0) \to (\mathbb{R}^{2n-1}, \bar{\Lambda}_0)$,*

$$(q^1, \ldots, q^n, p_1, \ldots, p_n) \mapsto (c_1, \ldots, c_{n-1}, d_1, \ldots, d_n) ,$$

is a Poisson morphism.

Proof.

A simple computation leads to the required results. Relations (i) and (ii) prove the existence of a bi-Hamiltonian system on $\mathbb{R}^{2n}$, while (iii) ensures that Λ_0 reduces to $\bar{\Lambda}_0$. □

Since Λ_0 is non-degenerate, we can define a recursion operator, by setting $R = \Lambda_1^\# \circ (\Lambda_0^\#)^{-1}$. We get

$$R = \begin{bmatrix} A & B \\ C & A^T \end{bmatrix}$$

where $A = [a_{ij}]$, $B = [b_{ij}]$ and $C = [c_{ij}]$ are $n \times n$ matrices (A^T is the transpose of A), that are defined as follows (with the convention that $q^{n+1} = +\infty$):

$$\begin{cases} a_{ii} = \exp(q^i - q^{i+1} + p_i) & \\ a_{i,i+1} = \exp(q^{i+1} - q^{i+2} + p_{i+1}) & \\ a_{ij} = 0, & \text{if } i > j \\ a_{ij} = \exp(q^j - q^{j+1} + p_j) - \exp(q^{j-1} - q^j + p_{j-1}), & \text{if } j > i+1; \end{cases}$$

$$\begin{cases} b_{ij} = -b_{ji} & \\ b_{ij} = \exp(q^j - q^{j+1} + p_j) + \exp(p_j), & \text{if } i < j \end{cases}$$

and

$$\begin{cases} c_{i+1,i} = -c_{i,i+1} = \exp(q^i - q^{i+1} + p_i) & \\ c_{ij} = 0, & \text{otherwise.} \end{cases}$$

Once we have a recursion operator R, an infinite sequence of pairwise compatible Poisson tensors on $\mathbb{R}^{2n}$, $\Lambda_k = R^k \Lambda_0$, and an infinite sequence of Hamiltonians H_k given by $dH_k = {}^tR(dH_{k-1})$ are then defined. By the reduction theorem of bi-Hamiltonian manifolds [13], and taking account of 1.1(iii), the infinite sequence $(\Lambda_k), k \in \mathbb{N}_0$, of higher order Poisson tensors on $\mathbb{R}^{2n}$ reduce, by π, to an infinite sequence $(\bar{\Lambda}_k), k \in \mathbb{N}_0$, of pairwise compatible Poisson tensors on $\mathbb{R}^{2n-1}$.

For $i = 2$, the Poisson tensor Λ_2 is given by

$$\begin{aligned} \Lambda_2 \quad = \quad & \sum_{j=1}^{n-2} \frac{\partial}{\partial q^j} \wedge \Bigg(\exp(q^{j+1} - q^{j+2} + p_{j+1}) \frac{\partial}{\partial p_{j+1}} \\ & + \sum_{i=j+1}^{n-1} \Bigg(\Big(- \exp(q^{i+1} - q^{i+2} + p_{i+1}) - \exp(p_{i+1}) \Big) \frac{\partial}{\partial q^i} \\ & + \Big(\exp(q^{i+1} - q^{i+2} + p_{i+1}) - \exp(q^i - q^{i+1} + p_i) \Big) \frac{\partial}{\partial p_{i+1}} \Bigg) \Bigg) \\ & + \sum_{i=1}^{n} \Bigg(\Big(\exp(q^i - q^{i+1} + p_i) + \exp(p_i) \Big) \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i} \\ & - \exp(q^i - q^{i+1} + p_i) \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial p_{i+1}} \Bigg) , \end{aligned}$$

and the corresponding reduced Poisson tensor $\bar{\Lambda}_2$ on $\mathbb{R}^{2n-1}$ is the one associated with the cubic bracket that appears in [6] and also in [4].

2 Master symmetries for the relativistic Toda lattice

Now, we want to find master symmetries for the bi-Hamiltonian system built above, in order to use the method of R. L. Fernandes [7], which is based on the following theorem.

Theorem 2.1 (Oevel) *Let X_0 be a vector field on the Poisson-Nijenhuis manifold $(M, \Lambda_0, \Lambda_1)$, such that*

$$\mathcal{L}(X_0)\Lambda_0 = \alpha\Lambda_0, \quad \mathcal{L}(X_0)\Lambda_1 = \beta\Lambda_1 \quad \textit{and} \quad X_0.H_0 = \gamma H_0, \tag{10}$$

with $\alpha, \beta, \gamma \in \mathbb{R}$. Then the vector fields $X_k = R^k X_0$ satisfy, for all $k, l \in \mathbb{N}$,

(i) $[X_k, X_l] = (\beta - \alpha)(l - k)X_{k+l}$;

(ii) $[X_k, Y_l] = (\beta + \gamma + (\beta - \alpha)(l - 1))Y_{k+l}$;

(iii) $\mathcal{L}(X_k)\Lambda_l = (\beta + (\beta - \alpha)(l - k - 1))\Lambda_{k+l}$;

(iv) $X_k.H_l = (\gamma + (\beta - \alpha)(l + k))H_{k+l}$.

We have to define a vector field that satisfies conditions (10) of theorem 2.1. We take

$$X_0 = \sum_{i=1}^{n} \frac{\partial}{\partial p_i}, \tag{11}$$

and we compute

$$\mathcal{L}(X_0)\Lambda_0 = -\Lambda_0, \quad \mathcal{L}(X_0)\Lambda_1 = 0 \quad \text{and} \quad X_0.H_0 = H_0.$$

So theorem 2.1 holds with $\alpha = -1, \beta = 0$ and $\gamma = 1$ and we have a hierarchy of master symmetries $X_k = R^k X_0, k \in \mathbb{N}$, that provides a way of getting higher order Poisson tensors on $\mathbb{R}^{2n}$. These master symmetries satisfy the following conditions:

$$\textbf{a)} \quad [X_k, X_l] = (l - k)X_{k+l}; \tag{12}$$

$$\textbf{b)} \quad [X_k, \Lambda_l^{\#}(dH_0)] = l\Lambda_{k+l}^{\#}(dH_0); \tag{13}$$

$$\textbf{c)} \quad \mathcal{L}(X_k)\Lambda_l = (l - k - 1)\Lambda_{k+l}; \tag{14}$$

$$\textbf{d)} \quad X_k.H_l = (1 + l + k)H_{k+l}. \tag{15}$$

Proposition 2.1 *The master symmetries $X_k = R^k X_0, k \in \mathbb{N}$, are projectable vector fields by $\pi : \mathbb{R}^{2n} \to \mathbb{R}^{2n-1}, \quad (q^i, p_i) \mapsto (c_i, d_i)$. We denote by $\bar{X}_k$ the projected vector fields.*

Proof.
The fibres of π are integral curves of the vector field

$$Z = \sum_{i=1}^{n} \frac{\partial}{\partial q^i}.$$

Since $Z = -\sum_{i=1}^{n} \Lambda_1^{\#}(dp_i)$, this implies $[\Lambda_1, Z] = 0$.

Further, we compute $[\Lambda_0, Z] = 0$ and therefore $[R, Z] = 0$. Also, $[X_0, Z] = 0$ and we deduce

$$[X_k, Z] = [R^k X_0, Z] = 0.$$

For any $f \in C^\infty(\mathbb{R}^{2n}, \mathbb{R})$, we have

$$\begin{aligned}[][X_k, fZ] &= (X_k.f)Z + f[X_k, Z] \\ &= (X_k.f)Z\end{aligned}$$

which proves that X_k is a projectable vector field. □

Now, if we take the reduced vector fields $\bar{X}_k$, the reduced Poisson tensors $\bar{\Lambda}_k$, the reduced Hamiltonian vector fields $\bar{Y}_k = \bar{\Lambda}_k^\#(d\bar{H}_0)$ and the reduced Hamiltonians $\bar{H}_k$ then, taking account of relations (12), (13), (14) and (15), we deduce the following relations:

$$\mathbf{a)}\quad [\bar{X}_k, \bar{X}_l] = (l-k)\bar{X}_{k+l}; \tag{16}$$
$$\mathbf{b)}\quad [\bar{X}_k, \bar{Y}_l] = l\bar{Y}_{k+l}; \tag{17}$$
$$\mathbf{c)}\quad \mathcal{L}(\bar{X}_k)\bar{\Lambda}_l = (l-k-1)\bar{\Lambda}_{k+l}; \tag{18}$$
$$\mathbf{d)}\quad \bar{X}_k.\bar{H}_l = (1+l+k)\bar{H}_{k+l}. \tag{19}$$

Some of these relations already appeared in [6], although our vector field $\bar{X}_1$ differs from the corresponding one in [6] - let us denote it by $\dot{X}_1$, by a bi-Hamiltonian vector field. In fact, we compute

$$\begin{aligned} X_1 &= \sum_{i=1}^{n} \Big((1-i)\Big(\exp(q^i - q^{i+1} + p_i) + \exp(p_i)\Big) \\ &\quad + \sum_{j=i+1}^{n} \Big(\exp(q^j - q^{j+1} + p_j) + \exp(p_j)\Big)\Big) \frac{\partial}{\partial q^i} \\ &\quad + \sum_{i=1}^{n} \Big(\exp(p_i) + i\, \exp(q^i - q^{i+1} + p_i) \\ &\quad + (2-i)\exp(q^{i-1} - q^i + p_{i-1})\Big) \frac{\partial}{\partial p_i}, \end{aligned}$$

where, by convention, $q^0 = -\infty$ and $q^{n+1} = +\infty$, and X_1 projects onto

$$\begin{aligned} \bar{X}_1 &= \sum_{i=1}^{n} \Bigg(\Big((1+i)c_i(c_{i+1} + d_{i+1}) + (2-i)c_i(c_{i-1} + d_i) + c_i^2 \Big) \frac{\partial}{\partial c_i} \\ &\quad + \Big(i\, c_i d_i + (2-i)c_{i-1}d_i + d_i^2\Big) \frac{\partial}{\partial d_i} \Bigg), \end{aligned}$$

where, by convention, $c_0 = c_n = 0$. Comparing $\bar{X}_1$ and $\dot{X}_1$, we obtain

$$\dot{X}_1 - \bar{X}_1 = \bar{\Lambda}_1^\#(d\bar{H}_0) = \bar{\Lambda}_0^\#(d\bar{H}_1).$$

Since the difference is the Hamiltonian vector field $\bar{\Lambda}_1^\#(d\bar{H}_0)$, our higher order Poisson tensors $\bar{\Lambda}_k$ coincide with those of [6].

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