From momentum maps and dual pairs to symplectic and Poisson groupoids

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It is a great pleasure to submit a contribution for this volume in honour of Alan Weinstein. He is one of the four or five persons whose works have had the greatest influence on my own scientific interests, and I am glad to have this opportunity to express to him my admiration and my thanks.

Introduction

In this survey, we will try to indicate some important ideas, due in large part to Alan Weinstein, which led from the study of momentum maps and dual pairs to the current interest in symplectic and Poisson groupoids. We hope that it will be useful for readers new to the subject; therefore, we begin by recalling the definitions and properties which will be used in what follows. More details can be found in [4, 26, 47].

1. A *Poisson manifold* [27, 48] is a smooth manifold P equipped with a bivector field (i.e., a smooth section of $\bigwedge^2 TP$) Π which satisfies

$$[\Pi, \Pi] = 0,$$

the bracket on the left-hand side being the Schouten bracket [44, 40]. The bivector field Π will be called the *Poisson structure* on P. It allows us to define a composition law on the space $C^\infty(P, \mathbb{R})$ of smooth functions on P, called the *Poisson bracket* and denoted by $(f, g) \mapsto \{f, g\}$, by setting, for all f and $g \in C^\infty(P, \mathbb{R})$ and $x \in P$,

$$\{f, g\}(x) = \Pi\bigl(df(x), dg(x)\bigr).$$

That composition law is skew-symmetric and satisfies the Jacobi identity and therefore turns $C^\infty(P, \mathbb{R})$ into a Lie algebra.

2. Let (P, Π) be a Poisson manifold. We denote by $\Pi^\sharp : T^*P \to TP$ the vector bundle map defined by

$$\langle \eta, \Pi^\sharp(\zeta) \rangle = \Pi(\zeta, \eta),$$

where ζ and η are two elements in the same fibre of T^*P. Let $f : P \to \mathbb{R}$ be a smooth function on P. The vector field $X_f = \Pi^\sharp(df)$ is called the *Hamiltonian vector field* associated to f. If $g : P \to \mathbb{R}$ is another smooth function on P, the Poisson bracket $\{f, g\}$ can be written

$$\{f, g\} = \langle dg, \Pi^\sharp(df) \rangle = -\langle df, \Pi^\sharp(dg) \rangle.$$

3. Every symplectic manifold (M, ω) has a Poisson structure, associated to its symplectic structure, for which the vector bundle map $\Pi^\sharp : T^*M \to M$ is the inverse of the vector bundle isomorphism $v \mapsto -i(v)\omega$. We will always consider that a symplectic manifold is equipped with that Poisson structure, unless otherwise specified.

4. Let (P_1, Π_1) and (P_2, Π_2) be two Poisson manifolds. A smooth map $\varphi : P_1 \to P_2$ is called a *Poisson map* if for every pair (f, g) of smooth functions on P_2,

$$\{\varphi^* f, \varphi^* g\}_1 = \varphi^*\{f, g\}_2.$$

5. The product $P_1 \times P_2$ of two Poisson manifolds (P_1, Π_1) and (P_2, Π_2) has a natural Poisson structure: it is the unique Poisson structure for which the bracket of functions of the form $(x_1, x_2) \mapsto f_1(x_1) f_2(x_2)$ and $(x_1, x_2) \mapsto g_1(x_1) g_2(x_2)$, where f_1 and $g_1 \in C^\infty(P_1, \mathbb{R})$, f_2 and $g_2 \in C^\infty(P_2, \mathbb{R})$, is $(x_1, x_2) \mapsto \{f_1, g_1\}_1(x_1)\{f_2, g_2\}_2(x_2)$. The same property holds for the product of any finite number of Poisson manifolds.

6. Let (V, ω) be a symplectic vector space, which means a real, finite-dimensional vector space V with a skew-symmetric nondegenrate bilinear form ω. Let W be a vector subspace of V. The *symplectic orthogonal* of W is

$$\operatorname{orth} W = \{v \in V;\, \omega(v, w) = 0 \text{ for all } w \in W\}.$$

It is a vector subspace of V, which satisfies

$$\dim W + \dim(\operatorname{orth} W) = \dim V, \quad \operatorname{orth}(\operatorname{orth} W) = W.$$

The vector subspace W is said to be *isotropic* if $W \subset \operatorname{orth} W$, *coisotropic* if $\operatorname{orth} W \subset W$, and *Lagrangian* if $W = \operatorname{orth} W$. In any symplectic vector space, there are many Lagrangian subspaces; therefore, the dimension of a symplectic vector space is always even; if $\dim V = 2n$, the dimension of an isotropic (resp., coisotropic, resp., Lagrangian) vector subspace is $\leq n$ (resp., $\geq n$, resp., $= n$).

7. A submanifold N of a Poisson manifold (P, Π) is said to be *coisotropic* if the bracket of two smooth functions, defined on an open subset of P and which vanish on N, vanishes on N too. A submanifold N of a symplectic manifold (M, ω) is coisotropic if and only if for each point $x \in N$, the vector subspace $T_x N$ of the symplectic vector space $(T_x M, \omega(x))$ is coisotropic. Therefore, the dimension of a coisotropic submanifold in a $2n$-dimensional symplectic manifold is $\geq n$; when it is equal to n, the submanifold N is said to be *Lagrangian*.

8. A *dual pair* [48, 4] is a pair $(\varphi_1 : M \to P_1, \varphi_2 : M \to P_2)$ of smooth Poisson maps, defined on the same symplectic manifold (M, ω), with values in the Poisson manifolds (P_1, Π_1) and (P_2, Π_2), such that for each $x \in M$, the two equivalent equalities hold:

$$\ker(T_x\varphi_1) = \operatorname{orth}\big(\ker(T_x\varphi_2)\big), \quad \ker(T_x\varphi_2) = \operatorname{orth}\big(\ker(T_x\varphi_1)\big).$$

That property implies that for all $f_1 \in C^\infty(P_1, \mathbb{R})$ and $f_2 \in C^\infty(P_2, \mathbb{R})$,

$$\{\varphi_1^* f_1, \varphi_2^* f_2\} = 0.$$

A dual pair (φ_1, φ_2) is specially interesting when φ_1 and φ_2 are surjective submersions.

9. Let $\varphi : M \to P$ be a surjective submersion of a symplectic manifold (M, ω) onto a manifold P. The manifold P has a Poisson structure Π for which φ is a Poisson map if and only if $\operatorname{orth}(\ker T\varphi)$ is integrable [23]. When that condition is satisfied, this Poisson structure on P is unique. If in addition there exist a smooth manifold P_2 and a smooth map $\varphi_2 : M \to P_2$ such that $\ker T\varphi_2 = \operatorname{orth}(\ker T\varphi)$, the manifold P_2 has a unique Poisson structure Π_2 for which φ_2 is a Poisson map, and (φ, φ_2) is a dual pair.

10. A *symplectic realization* [48] of a Poisson manifold (P, Π) is a Poisson map $\varphi : M \to P$, defined on a symplectic manifold (M, ω), with values in P. A symplectic realization $\varphi : M \to P$ is specially interesting when φ is a surjective submersion; such a symplectic realization is said to be *surjective submersive*.

For example, in a dual pair $(\varphi_1 : M \to P_1, \varphi_2 : M \to P_2)$, φ_1 and φ_2 are symplectic realizations of the Poisson manifolds (P_1, Π_1) and (P_2, Π_2), respectively.

1 A typical example: action of a Lie group on its cotangent bundle

Let G be a Lie group, $\pi_G : T^*G \to G$ its cotangent bundle. For each g and $h \in G$, we write

$$L_g(h) = gh, \quad R_g(h) = hg.$$

We denote by $\widehat{L}_g$ and $\widehat{R}_g$ the canonical lifts to T^*G of L_g and R_g, respectively. We recall that, for g and $h \in G$, $\xi \in T_h^*G$, $X \in T_{gh}G$ and $Y \in T_{hg}G$, we have

$$\big\langle \widehat{L}_g(\xi), X \big\rangle = \langle \xi, TL_{g^{-1}} X \rangle, \quad \big\langle \widehat{R}_g(\xi), Y \big\rangle = \langle \xi, TR_{g^{-1}} Y \rangle.$$

The maps $\widehat{L} : G \times T^*G \to T^*G$, $(g, \xi) \mapsto \widehat{L}_g\xi$, and $\widehat{R} : T^*G \times G \to T^*G$, $(\xi, g) \mapsto \widehat{R}_g\xi$, are actions of the Lie group G on its cotangent bundle T^*G, on the left and on the right, respectively.

Theorem 1.1. *These two actions are Hamiltonian and have as momentum maps, respectively, the maps*

$$J_L : T^*G \to \mathcal{G}^*, \qquad J_L(\xi) = \widehat{R}_{\pi_G(\xi)^{-1}}\xi$$

and

$$J_R : T^*G \to \mathcal{G}^*, \qquad J_R(\xi) = \widehat{L}_{\pi_G(\xi)^{-1}}\xi,$$

where $\mathcal{G}^*$ is the dual of the Lie algebra $\mathcal{G}$ of G. We have identified $\mathcal{G}$ with T_eG and $\mathcal{G}^*$ with T^*_eG. We have denoted by $\pi_G : T^*G \to G$ the cotangent bundle projection. Moreover,

- *the level sets of J_L are the orbits of the action $\widehat{R}$,*
- *the level sets of J_R are the orbits of the action $\widehat{L}$,*
- *for each $\xi \in T^*G$, each one of the tangent spaces at ξ to the orbits of that point under the actions $\widehat{R}$ and $\widehat{L}$ is the symplectic orthogonal of the other one.*

The dual $\mathcal{G}^*$ of the Lie algebra $\mathcal{G}$ has two (opposite) natural Poisson structures, called the *plus* and the *minus KKS-Poisson structures* (the letters KKS stand for Kirillov [19], Kostant [21], and Souriau [45]). The brackets of two smooth functions f and $g \in C^\infty(\mathcal{G}^*, \mathbb{R})$ are given by the formulae

$$\{f, g\}_+(\eta) = \big\langle \eta, \big[df(\eta), dg(\eta)\big]\big\rangle \quad \text{and} \quad \{f, g\}_-(\eta) = -\big\langle \eta, \big[df(\eta), dg(\eta)\big]\big\rangle.$$

Then we have the following.

Theorem 1.2. *The map $J_L : T^*G \to \mathcal{G}^*$ (resp., the map $J_R : T^*G \to \mathcal{G}^*$) is a Poisson map when T^*G is equipped with the Poisson structure associated to its canonical symplectic structure and $\mathcal{G}^*$ with the minus KKS-Poisson structure (resp, with the plus KKS-Poisson structure).*

The pair of maps (J_L, J_R) in the above theorem is a very simple example of a *dual pair*; both J_L and J_R are surjective submersions. The cotangent space T^*G, equipped with its canonical symplectic structure, is a *symplectic realization* of each of the Poisson manifolds $\big(\mathcal{G}^*, \{,\}_+\big)$ and $\big(\mathcal{G}^*, \{,\}_-\big)$.

2 Hamiltonian action of a Lie group on a symplectic manifold

The example dealt with in the preceding section is very symmetrical: the roles of the two actions $\widehat{L}$ and $\widehat{R}$ could be exchanged by means of the group anti-automorphism $g \mapsto g^{-1}$ of the Lie group G. In this section, we will see that a similar situation (although less symmetrical) holds for a Hamiltonian action of a Lie group on a symplectic manifold.

Definitions 2.1. *Let $\Phi : G \times M \to M$ be an action (for example, on the left) of a Lie group G on a symplectic manifold (M, ω). That action is said to be* symplectic *if, for any $g \in G$,*

$$\Phi_g^*\omega = \omega.$$

It is said to be Hamiltonian *if it is symplectic and if, in addition, there exists a smooth map $J : M \to \mathcal{G}^*$ such that, for each $X \in \mathcal{G}$, the fundamental vector field X_M is Hamiltonian, with the function $\langle J, X\rangle : x \mapsto \big\langle J(x), X\big\rangle$ as Hamiltonian. The map J is called the* momentum map *of Φ.*

2.2 Comments

Let us indicate briefly some important properties of Hamiltonian actions.

The fundamental vector fields

Under the assumptions of the above definitions, we recall that the *fundamental vector field* X_M associated to an element $X \in \mathcal{G}$ (the Lie algebra of G) is the vector field on M defined by

$$X_M(x) = \frac{d}{dt}\Phi\bigl(\exp(tX), x\bigr)\Big|_{t=0},$$

where $x \in M$. The formula

$$i(X_M)\omega(x) = -d\langle J, X\rangle(x)$$

expresses that X_M is the Hamiltonian vector field associated to the function $\langle J, X\rangle$.

The momentum map equivariance

When M is connected, there exists a smooth map $\theta : G \to \mathcal{G}^*$ such that, for all $g \in G$ and $x \in M$,

$$J\bigl(\Phi(g, x)\bigr) = \mathrm{Ad}^*_g\bigl(J(x)\bigr) + \theta(g),$$

where $\mathrm{Ad}^* : G \times \mathcal{G}^* \to \mathcal{G}^*$ is the coadjoint action, defined by

$$\bigl\langle \mathrm{Ad}^*_g(\xi), X\bigr\rangle = \bigl\langle \xi, \mathrm{Ad}_{g^{-1}}(X)\bigr\rangle, \quad \xi \in \mathcal{G}^*, \quad X \in \mathcal{G}.$$

The map θ satisfies, for all g and $h \in G$,

$$\theta(gh) = \mathrm{Ad}^*_g\bigl(\theta(h)\bigr) + \theta(g),$$

and is called a *symplectic cocycle of* G [45]. As a consequence, the map

$$a_\theta : G \times \mathcal{G}^* \to \mathcal{G}^*, \quad a_\theta(g, \xi) = \mathrm{Ad}^*_g(\xi) + \theta(g),$$

is an action on the left of G on $\mathcal{G}^*$, called the *affine action associated to* θ. The momentum map $J : M \to \mathcal{G}^*$ is equivariant when G acts on M by the action Φ and on $\mathcal{G}^*$ by the affine action a_θ.

The modified KKS-Poisson structures

Moreover, the momentum map J is a Poisson map when M is equipped with the Poisson structure associated to its symplectic structure, and $\mathcal{G}^*$ with the modified KKS-Poisson structure [45]:

$$\{f, g\}^\theta_-(\xi) = -\bigl\langle \xi, \bigl[df(\xi), dg(\xi)\bigr]\bigr\rangle + \Theta\bigl(df(\xi), dg(\xi)\bigr),$$

where $\Theta : \mathcal{G} \times \mathcal{G} \to \mathbb{R}$ is the map defined by

$$\Theta(X, Y) = \big\langle T_e\theta(X), Y\big\rangle.$$

When considered as a linear map from $\mathcal{G}$ to its dual $\mathcal{G}^*$, the map Θ is a $\mathcal{G}^*$-valued 1-cocycle of the Lie algebra $\mathcal{G}$, which corresponds to the 1-cocycle θ of the Lie group G. Considered as a real-valued skew-symmetric bilinear map on $\mathcal{G} \times \mathcal{G}$, Θ is a real-valued 2-cocycle of the Lie algebra $\mathcal{G}$.

Symplectic orthogonality

For each $x \in M$, each one of the vector subspaces of the symplectic vector space $\big(T_x M, \omega(x)\big)$,

- the tangent space at x to the orbit $\Phi(G, x)$,
- the kernel of $T_x J$,

is the symplectic orthogonal of the other one.

2.3 Remarks and questions

Several features of dual pairs and symplectic realizations exist in a Hamiltonian action of a Lie group G on a symplectic manifold (M, ω): the momentum map $J : M \to \mathcal{G}^*$ is a symplectic realization of $\mathcal{G}^*$ equipped with the modified KKS-Poisson structure; the symplectic orthogonality, for all $x \in M$, of $\ker T_x J$ and $T_x\Phi(G, x)$, looks very much like the property of a dual pair; more precisely, if we assume that the set M/G of orbits of Φ has a smooth manifold structure for which the canonical projection $\pi : M \to M/G$ is a submersion, there is on M/G a unique Poisson structure for which π is a Poisson map, and (J, π) is a dual pair. However, the momentum map J may not be a submersion (it is a submersion if and only if the action Φ is locally free), and the set M/G may not have a smooth manifold structure for which π is a submersion.

In order to make the Hamiltonian action of a Lie group more like the simple example of Section 1, we are led to the following questions:

1. Does there exist an "action" (in a generalized sense) on the manifold M whose orbits are the level sets of the momentum map J?
2. In the theory of momentum maps, can we replace the dual of a Lie algebra by a more general Poisson manifold?
3. Does any Poisson manifold have a symplectic realization?

The notion of symplectic groupoid, introduced in the next section, will allow us to answer some of these questions.

3 Lie groupoids and symplectic groupoids

We recall below the definition of a groupoid. The reader will find examples and more information about groupoids in [4, 6, 33, 43, 52].

Definition 3.1. *A* groupoid *is a set* Γ *equipped with the structure defined by the following data:*

- *a subset* Γ_0 *of* Γ*, called the* set of units *of the groupoid;*
- *two maps* $\alpha : \Gamma \to \Gamma_0$ *and* $\beta : \Gamma \to \Gamma_0$*, called, respectively, the* target map *and the* source map*; they satisfy*

$$\alpha|_{\Gamma_0} = \beta|_{\Gamma_0} = \mathrm{id}_{\Gamma_0};$$

- *a composition law* $m : \Gamma_2 \to \Gamma$*, called the* product*, defined on the subset* Γ_2 *of* $\Gamma \times \Gamma$*,*

$$\Gamma_2 = \big\{(x, y) \in \Gamma \times \Gamma;\, \beta(x) = \alpha(y)\big\},$$

  which is associative, in the sense that whenever one side of the equality

$$m\big(x, m(y, z)\big) = m\big(m(x, y), z\big)$$

  is defined, the other side is defined too, and the equality holds; moreover, the composition law m *is such that for each* $x \in \Gamma$*,*

$$m\big(\alpha(x), x\big) = m\big(x, \beta(x)\big) = x;$$

- *a map* $\iota : \Gamma \to \Gamma$*, called the* inverse*, such that, for every* $x \in \Gamma$*,* $\big(x, \iota(x)\big) \in \Gamma_2$ *and* $\big(\iota(x), x\big) \in \Gamma_2$*, and*

$$m\big(x, \iota(x)\big) = \alpha(x), \qquad m\big(\iota(x), x\big) = \beta(x).$$

3.2 Properties and comments

The above definitions have the following consequences.

Involutivity of the inverse

The inverse map ι is involutive. We have indeed, for any $x \in \Gamma$,

$$\begin{aligned} i^2(x) &= m\big(i^2(x), \beta\big(i^2(x)\big)\big) = m\big(i^2(x), \beta(x)\big) = m\big(i^2(x), m\big(i(x), x\big)\big) \\ &= m\big(m\big(i^2(x), i(x)\big), x\big) = m\big(\alpha(x), x\big) = x. \end{aligned}$$

Unicity of the inverse

Let x and $y \in \Gamma$ be such that

$$m(x, y) = \alpha(x) \quad \text{and} \quad m(y, x) = \beta(x).$$

Then we have

$$\begin{aligned} y &= m\big(y, \beta(y)\big) = m\big(y, \alpha(x)\big) = m\big(y, m\big(x, \iota(x)\big)\big) = m\big(m(y, x), \iota(x)\big) \\ &= m\big(\beta(x), \iota(x)\big) = m\big(\alpha\big(\iota(x)\big), \iota(x)\big) = \iota(x). \end{aligned}$$

Therefore, for any $x \in \Gamma$, the unique $y \in \Gamma$ such that $m(y, x) = \beta(x)$ and $m(x, y) = \alpha(x)$ is $\iota(x)$.

Notations

A groupoid Γ with set of units Γ_0 and target and source maps α and β will be denoted by $\Gamma \overset{\alpha}{\underset{\beta}{\rightrightarrows}} \Gamma_0$.

When there is no risk of error, for x and $y \in \Gamma$, we will write $x.y$, or even simply xy for $m(x,y)$, and x^{-1} for $\iota(x)$.

Definitions 3.3. *A* topological groupoid *is a groupoid* $\Gamma \overset{\alpha}{\underset{\beta}{\rightrightarrows}} \Gamma_0$ *for which* Γ *is a (maybe non-Hausdorff) topological space,* Γ_0 *a Hausdorff topological subspace of* Γ*,* α *and* β *surjective continuous maps,* $m : \Gamma_2 \to \Gamma$ *a continuous map and* $\iota : \Gamma \to \Gamma$ *an homeomorphism.*

A Lie groupoid *is a groupoid* $\Gamma \overset{\alpha}{\underset{\beta}{\rightrightarrows}} \Gamma_0$ *for which* Γ *is a smooth (maybe non-Hausdorff) manifold,* Γ_0 *a smooth Hausdorff submanifold of* Γ*,* α *and* β *smooth surjective submersions (which implies that* Γ_2 *is a smooth submanifold of* $\Gamma \times \Gamma$*),* $m : \Gamma_2 \to \Gamma$ *a smooth map and* $\iota : \Gamma \to \Gamma$ *a smooth diffeomorphism.*

3.4 Properties and examples of Lie groupoids

1. Let $\Gamma \overset{\alpha}{\underset{\beta}{\rightrightarrows}} \Gamma_0$ be a Lie groupoid. Since α and β are submersions, for any $x \in \Gamma$, $\alpha^{-1}\bigl(\alpha(x)\bigr)$ and $\beta^{-1}\bigl(\beta(x)\bigr)$ are submanifolds of Γ, both of dimension $\dim\Gamma - \dim\Gamma_0$, called the α*-fibre* and the β*-fibre* through x, respectively. The inverse map ι, restricted to the α-fibre through x (resp., the β-fibre through x) is a diffeomorphism of that fibre onto the β-fibre through $\iota(x)$ (resp., the α-fibre through $\iota(x)$). The dimension of the submanifold Γ_2 of composable pairs in $\Gamma \times \Gamma$ is $2\dim\Gamma - \dim\Gamma_0$.

2. A Lie group is a Lie groupoid whose set of units has only one element e.

3. A vector bundle $\pi : E \to M$ is a Lie groupoid, with the base M as a set of units; the source and target maps both coincide with the projection π, the product and the inverse maps are the addition $(x,y) \mapsto x+y$ and the opposite map $x \mapsto -x$ in the fibres.

4. Let $\Gamma \overset{\alpha}{\underset{\beta}{\rightrightarrows}} \Gamma_0$ be a Lie groupoid. Its tangent bundle $T\Gamma$ is a Lie groupoid, with $T\Gamma_0$ as a set of units, $T\alpha : T\Gamma \to T\Gamma_0$, and $T\beta : T\Gamma \to T\Gamma_0$ as target and source maps. Let us denote by Γ_2 the set of composable pairs in $\Gamma \times \Gamma$, by $m : \Gamma_2 \to \Gamma$ the composition law and by $\iota : \Gamma \to \Gamma$ the inverse. Then the set of composable pairs in $T\Gamma \times T\Gamma$ is simply $T\Gamma_2$, the composition law on $T\Gamma$ is $Tm : T\Gamma_2 \to T\Gamma$ and the inverse is $T\iota : T\Gamma \to T\Gamma$.

When the groupoid Γ is a Lie group G, the Lie groupoid TG is a Lie group too.

We will see below that the cotangent bundle of a Lie groupoid is a Lie groupoid, and more precisely a symplectic groupoid.

Let us now introduce the important notion of symplectic groupoid, first considered by A. Weinstein [6], M. Karasev [18] and S. Zakrzewski [58].

Definition 3.5. *A* symplectic groupoid *is a Lie groupoid* $\Gamma \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \Gamma_0$ *with a symplectic form* ω *on* Γ *such that the graph of the composition law* m

$$\bigl\{(x, y, z) \in \Gamma \times \Gamma \times \Gamma;\ (x, y) \in \Gamma_2 \text{ and } z = m(x, y)\bigr\}$$

is a Lagrangian submanifold of $\Gamma \times \Gamma \times \overline{\Gamma}$ *with the product symplectic form, the first two factors* Γ *being endowed with the symplectic form* ω*, and the third factor* $\overline{\Gamma}$ *being* Γ *with the symplectic form* $-\omega$*.*

The next theorem states important properties of symplectic groupoids.

Theorem 3.6. *Let* $\Gamma \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \Gamma_0$ *be a symplectic groupoid, with the symplectic* 2*-form* ω*. We have the following properties:*

1. *For any point* $c \in \Gamma$*, each one of the two vector subspaces of the symplectic vector space* $\bigl(T_c\Gamma, \omega(c)\bigr)$*,*

$$T_c\bigl(\beta^{-1}\bigl(\beta(c)\bigr)\bigr) \quad \text{and} \quad T_c\bigl(\alpha^{-1}\bigl(\alpha(c)\bigr)\bigr)$$

   is the symplectic orthogonal of the other one.

2 *The submanifold of units* Γ_0 *is a Lagrangian submanifold of the symplectic manifold* (Γ, ω)*.*

3. *The inverse map* $\iota : \Gamma \to \Gamma$ *is an antisymplectomorphism of* (Γ, ω) *(it satisfies* $\iota^*\omega = -\omega$*).*

Proof. For each $x \in \Gamma$, we denote by $P_x = \beta^{-1}\bigl(\beta(x)\bigr)$ and $Q_x = \alpha^{-1}\bigl(\alpha(x)\bigr)$ the β-fibre and the α-fibre through x, respectively. Let $\dim \Gamma = 2n$. We have seen that $\dim \Gamma_2 = 2 \dim \Gamma - \dim \Gamma_0$. Since the graph of the product is a Lagrangian submanifold of $\Gamma \times \Gamma \times \overline{\Gamma}$, its dimension is half that of $\Gamma \times \Gamma \times \overline{\Gamma}$, so

$$2(2 \dim \Gamma - \dim \Gamma_0) = 3 \dim \Gamma;$$

therefore, $\dim \Gamma_0 = (1/2) \dim \Gamma = n$. Since α and β are smooth submersions, for each point $x \in \Gamma$, P_x and Q_x are smooth n-dimensional submanifolds of Γ.

1. Let $(a, b) \in \Gamma_2$ and $c = m(a, b)$. The maps $L_a : y \mapsto m(a, y)$ and $R_b : x \mapsto m(x, b)$ are diffeomorphisms which map, respectively, Q_b onto Q_c and P_a onto P_c. Therefore, if $u \in T_a P_a$ and $v \in T_b Q_b$, w_1 and $w_2 \in T_c\Gamma$, the vectors $(u, 0, w_1)$ and $(0, v, w_2) \in T_{(a,b,c)}(\Gamma \times \Gamma \times \Gamma)$ are tangent to the graph of the product m if and only if

$$w_1 = T R_b(u) \quad \text{and} \quad w_2 = T L_a(v).$$

By writing that the graph of m is Lagrangian in $\Gamma \times \Gamma \times \overline{\Gamma}$, we obtain

$$\omega(u, 0) + \omega(0, v) - \omega\bigl(T R_b(u), T L_a(v)\bigr) = 0,$$

so, for all $u \in T_a P_a$ and $v \in T_b Q_b$,

$$\omega\big(TR_b(u), TL_a(v)\big) = 0.$$

Since $TR_b(u)$ can be any vector in T_cP_c and $TL_a(v)$ any vector in T_cQ_c, we have shown that $\text{orth}(T_cP_c) \supset T_cQ_c$. But since these two subspaces are of dimension n, $\text{orth}(T_cP_c) = T_cQ_c$, and $\text{orth}(T_cQ_c) = T_cP_c$.

2. Let $p \in \Gamma_0$. We have $m(p, p) = p$; therefore, for u_1 and $u_2 \in T_p\Gamma_0$, (u_1, u_1, u_1) and (u_2, u_2, u_2) are tangent to the graph of m. Since that graph is Lagrangian in $\Gamma \times \Gamma \times \overline{\Gamma}$, we have

$$0 = \omega(u_1, u_2) + \omega(u_1, u_2) - \omega(u_1, u_2) = \omega(u_1, u_2).$$

This shows that Γ_0 is an isotropic submanifold of Γ. But since $\dim \Gamma_0 = n$, Γ_0 is Lagrangian.

3. Let $x \in \Gamma$, u_1 and $u_2 \in T_x\Gamma$, $v_1 = T\iota(u_1)$, $v_2 = T\iota(u_2)$. Let $t \mapsto s_1(t)$ and $t \mapsto s_2(t)$ be smooth curves in Γ such that $s_1(0) = s_2(0) = x$, $\big(ds_1(t)/dt\big)|_{t=0} = u_1$ and $\big(ds_2(t)/dt\big)|_{t=0} = u_2$. For $i = 1$ or 2, we have

$$m\big(s_i(t), \iota \circ s_i(t)\big) = \alpha \circ s_i(x).$$

Therefore, if we set $w_1 = T\alpha(u_1)$ and $w_2 = T\alpha(u_2)$, the vectors (u_1, v_1, w_1) and (u_2, v_2, w_2) are tangent at $\big(x, \iota(x), \alpha(x)\big)$ to the graph of m. Since that graph is Lagrangian in $\Gamma \times \Gamma \times \overline{\Gamma}$, we have

$$\omega(u_1, u_2) + \omega\big(T\iota(u_1), T\iota(u_2)\big) - \omega(w_1, w_2) = 0.$$

But since w_1 and w_2 are tangent at $\alpha(x)$ to the Lagrangian submanifold Γ_0, $\omega(w_1, w_2) = 0$. It follows that $\iota : \Gamma \to \Gamma$ is an antisymplectomorphism. □

Corollary 3.7. *Let* $\Gamma \overset{\alpha}{\underset{\beta}{\rightrightarrows}} \Gamma_0$ *be a symplectic groupoid, with symplectic* 2*-form* ω. *There exists on* Γ_0 *a unique Poisson structure* Π *for which* $\alpha : \Gamma \to \Gamma_0$ *is a Poisson map, and* $\beta : \Gamma \to \Gamma_0$ *an anti-Poisson map* (*i.e.,* β *is a Poisson map when* Γ_0 *is equipped with the Poisson structure* $-\Pi$). *The pair* $\big(\alpha : (\Gamma, \omega) \to (\Gamma_0, \Pi), \beta : (\Gamma, \omega) \to (\Gamma_0, -\Pi)\big)$ *is a dual pair, and* (Γ, ω) *is a symplectic realization of both* (Γ_0, Π) *and* $(\Gamma_0, -\Pi)$.

Proof. According to the previous theorem, the symplectic orthogonal of $\ker(T\alpha)$ is $\ker(T\beta)$. By Property 9 of the Introduction, there exists on Γ_0 a unique Poisson structure Π such that α is a Poisson map from (Γ, ω) to (Γ_0, Π). For the same reason there exists on Γ_0 another unique Poisson structure Π' such that β is a Poisson map from (Γ, ω) to (Γ_0, Π'). But $\beta = \alpha \circ \iota$, and the previous theorem shows that ι is an antisymplectomorphism. Therefore, $\Pi' = -\Pi$. □

3.8 Cotangent bundle of a Lie groupoid

Let $\Gamma \overset{\alpha}{\underset{\beta}{\rightrightarrows}} \Gamma_0$ be a Lie groupoid. We have seen above that its tangent bundle $T\Gamma$ has a Lie groupoid structure, determined by that of Γ. Similarly (but much less obviously) the cotangent bundle $T^*\Gamma$ has a Lie groupoid structure determined by that of Γ. The set of units is the conormal bundle to the submanifold Γ_0 of Γ, denoted by $\mathcal{N}^*\Gamma_0$. We recall that $\mathcal{N}^*\Gamma_0$ is the vector subbundle of $T^*_{\Gamma_0}\Gamma$ (the restriction to Γ_0 of the cotangent bundle $T^*\Gamma$) whose fibre $\mathcal{N}^*_p\Gamma_0$ at a point $p \in \Gamma_0$ is

$$\mathcal{N}^*_p\Gamma_0 = \bigl\{\eta \in T^*_p\Gamma;\ \langle \eta, v\rangle = 0 \quad \text{for all } v \in T_p\Gamma_0\bigr\}.$$

To define the target and source maps of the Lie algebroid $T^*\Gamma$, we introduce the notion of *bisection* through a point $x \in \Gamma$ [6, 1, 4]. A bisection through x is a submanifold A of Γ, with $x \in A$, transverse both to the α-fibres and to the β-fibres, such that the maps α and β, when restricted to A, are diffeomorphisms of A onto open subsets $\alpha(A)$ and $\beta(A)$ of Γ_0, respectively. For any point $x \in M$, there exist bisections through x. A bisection A allows us to define two smooth diffeomorphisms between open subsets of Γ, denoted by L_A and R_A and called the *left* and *right translations by* A, respectively. They are defined by

$$L_A : \alpha^{-1}\bigl(\beta(A)\bigr) \to \alpha^{-1}\bigl(\alpha(A)\bigr), \qquad L_A(y) = m\bigl(\beta|_A^{-1} \circ \alpha(y), y\bigr),$$

and

$$R_A : \beta^{-1}\bigl(\alpha(A)\bigr) \to \beta^{-1}\bigl(\beta(A)\bigr), \qquad R_A(y) = m\bigl(y, \alpha|_A^{-1} \circ \beta(y)\bigr).$$

The definitions of the target and source maps for $T^*\Gamma$ rest on the following properties. Let x be a point in Γ and let A be a bisection through x. The two vector subspaces, $T_{\alpha(x)}\Gamma_0$ and $\ker T_{\alpha(x)}\beta$, are complementary in $T_{\alpha(x)}\Gamma$. For any $v \in T_{\alpha(x)}\Gamma$, $v - T\beta(v)$ is in $\ker T_{\alpha(x)}\beta$. Moreover, R_A maps the fibre $\beta^{-1}\bigl(\alpha(x)\bigr)$ onto the fibre $\beta^{-1}\bigl(\beta(x)\bigr)$, and its restriction to that fibre does not depend on the choice of A; its depends only on x. Therefore, $TR_A\bigl(v - T\beta(v)\bigr)$ is in $\ker T_x\beta$ and does not depend on the choice of A. We can define the map $\widehat{\alpha}$ by setting, for any $\xi \in T^*_x\Gamma$ and any $v \in T_{\alpha(x)}\Gamma$,

$$\bigl\langle \widehat{\alpha}(\xi), v\bigr\rangle = \bigl\langle \xi, TR_A\bigl(v - T\beta(v)\bigr)\bigr\rangle.$$

Similarly, we define $\widehat{\beta}$ by setting, for any $\xi \in T^*_x\Gamma$ and any $w \in T_{\beta(x)}\Gamma$,

$$\bigl\langle \widehat{\beta}(\xi), w\bigr\rangle = \bigl\langle \xi, TL_A\bigl(w - T\alpha(w)\bigr)\bigr\rangle.$$

We see that $\widehat{\alpha}$ and $\widehat{\beta}$ are unambiguously defined, smooth and take their values in the submanifold $\mathcal{N}^*\Gamma_0$ of $T^*\Gamma$. They satisfy

$$\pi_\Gamma \circ \widehat{\alpha} = \alpha \circ \pi_\Gamma, \qquad \pi_\Gamma \circ \widehat{\beta} = \beta \circ \pi_\Gamma,$$

where $\pi_\Gamma : T^*\Gamma \to \Gamma$ is the cotangent bundle projection.

Let us now define the composition law $\widehat{m}$ on $T^*\Gamma$. Let $\xi \in T^*_x\Gamma$ and $\eta \in T^*_y\Gamma$ be such that $\widehat{\beta}(\xi) = \widehat{\alpha}(\eta)$. That implies $\beta(x) = \alpha(y)$. Let A be a bisection through x and B a bisection through y. There exist a unique $\xi_{h\alpha} \in T^*_{\alpha(x)}\Gamma_0$ and a unique $\eta_{h\beta} \in T^*_{\beta(y)}\Gamma_0$ such that

$$\xi = (L_A^{-1})^*\bigl(\widehat{\beta}(\xi)\bigr) + \alpha_x^*\xi_{h\alpha}, \qquad \eta = (R_B^{-1})^*\bigl(\widehat{\alpha}(\xi)\bigr) + \beta_y^*\eta_{h\beta}.$$

Then $\widehat{m}(\xi, \eta)$ is given by

$$\widehat{m}(\xi, \eta) = \alpha^*_{xy}\xi_{h\alpha} + \beta^*_{xy}\eta_{h\beta} + (R_B^{-1})^*(L_A^{-1})^*\big(\widehat{\beta}(x)\big).$$

We observe that in the last term of the above expression we can replace $\widehat{\beta}(\xi)$ by $\widehat{\alpha}(\eta)$, since these two expressions are equal, and that $(R_B^{-1})^*(L_A^{-1})^* = (L_A^{-1})^*(R_B^{-1})^*$, since R_B and L_A commute.

Finally, the inverse $\widehat{\iota}$ in $T^*\Gamma$ is ι^*.

With its canonical symplectic form, $T^*\Gamma \overset{\widehat{\alpha}}{\underset{\widehat{\beta}}{\rightrightarrows}} \mathcal{N}^*\Gamma_0$ is a symplectic groupoid.

When the Lie groupoid Γ is a Lie group G, the Lie groupoid T^*G is not a Lie group, contrary to what happens for TG. Its set of units can be identified with $\mathcal{G}^*$, and we recover the typical example of Section 1, with $\widehat{\alpha} = J_L$ and $\widehat{\beta} = J_R$.

4 Lie algebroids

Let us now introduce the notion of a Lie algebroid, related to that of a Lie groupoid in the same way as the notion of a Lie algebra is related to that of a Lie group. That notion is due to J. Pradines [41].

Definition 4.1. *A* Lie algebroid *over a smooth manifold M is a smooth vector bundle $\pi : A \to M$ with base M, equipped with*

- *a composition law $(s_1, s_2) \mapsto \{s_1, s_2\}$ on the space $\Gamma^\infty(\pi)$ of smooth sections of π, called the* bracket*, for which that space is a Lie algebra,*
- *a vector bundle map $\rho : A \to TM$, over the identity map of M, called the* anchor map*, such that, for all s_1 and $s_2 \in \Gamma^\infty(\pi)$ and all $f \in C^\infty(M, \mathbb{R})$,*

$$\{s_1, fs_2\} = f\{s_1, s_2\} + \big((\rho \circ s_1).f\big)s_2.$$

4.2 Examples

1. A finite-dimensional Lie algebra is a Lie algebroid (with a base reduced to a point).

2. A tangent bundle $\tau_M : TM \to M$ to a smooth manifold M is a Lie algebroid, with the usual bracket of vector fields on M as composition law, and the identity map as anchor map. More generally, any integrable vector subbundle F of a tangent bundle $\tau_M : TM \to M$ is a Lie algebroid, still with the bracket of vector fields on M with values in F as composition law and the canonical injection of F into TM as anchor map.

3. Let $X : M \to TM$ be a smooth vector field on a smooth manifold M. By setting, for any pair (f, g) of smooth functions on M,

$$[f, g] = f(X.g) - g(X.f),$$

we define on $C^\infty(M,\mathbb{R})$ a composition law (which is a special case of a *Jacobi bracket*) which turns that space into a Lie algebra. We may identify the space $C^\infty(M,\mathbb{R})$ of smooth functions on M with the space of smooth sections of the trivial vector bundle $p_1 : M\times\mathbb{R}\to M$, where p_1 is the projection on the first factor; in that identification, a smooth function $f\in C^\infty(M,\mathbb{R})$ is identified with the section

$$s_f : M\to M\times\mathbb{R},\quad s_f(x)=\bigl(x, f(x)\bigr).$$

The Jacobi bracket of functions defined above becomes a bracket

$$(s_f, s_g)\mapsto \{s_f, s_g\}=s_{[f,g]}$$

on the space of smooth sections of the trivial bundle $p_1 : M\times\mathbb{R}\to M$. It is easy to see that with the anchor map

$$\rho : M\times\mathbb{R}\to TM,\quad \rho(x,k)=kX(x),$$

that bundle becomes a Lie algebroid over M.

4. Let (P,Π) be a Poisson manifold. Its cotangent bundle $\pi_P : T^*P\to P$ has a Lie algebroid structure, with $\Pi^\sharp : T^*P\to TP$ as anchor map. The composition law is the *bracket of* 1*-forms*, first obtained by Fuchssteiner [14], then independently by Magri and Morosi [36]. It will be denoted by $(\eta,\zeta)\mapsto[\eta,\zeta]$ (in order to avoid any confusion with the Poisson bracket of functions). It is given by the formula, in which η and ζ are 1-forms and X a vector field on P,

$$\bigl\langle[\eta,\zeta],X\bigr\rangle=\Pi\bigl(\eta,d\langle\zeta,X\rangle\bigr)+\Pi\bigl(d\langle\eta,X\rangle,\zeta\bigr)+\bigl(\mathcal{L}(X)\Pi\bigr)(\eta,\zeta).$$

We have denoted by $\mathcal{L}(X)\Pi$ the Lie derivative of the Poisson tensor Π with respect to the vector field X. Another equivalent formula for that composition law is

$$[\zeta,\eta]=\mathcal{L}(\Pi^\sharp\zeta)\eta-\mathcal{L}(\Pi^\sharp\eta)\zeta-d\bigl(\Pi(\zeta,\eta)\bigr).$$

The bracket of 1-forms is related to the Poisson bracket of functions by

$$[df,dg]=d\{f,g\}\quad\text{for all } f \text{ and } g\in C^\infty(P,\mathbb{R}).$$

4.3 Properties of Lie algebroids

Let $\pi : A\to M$ be a Lie algebroid with anchor map $\rho : A\to TM$.

1. It is easy to see that for any pair (s_1,s_2) of smooth sections of π,

$$\rho\circ\{s_1,s_2\}=[\rho\circ s_1,\rho\circ s_2],$$

which means that the map $s\mapsto\rho\circ s$ is a Lie algebra homomorphism from the Lie algebra of smooth sections of π into the Lie algebra of smooth vector fields on M.

2. The composition law $(s_1,s_2)\mapsto\{s_1,s_2\}$ on the space of sections of π extends into a composition law on the space of sections of exterior powers of (A,π,M), which will be called the *generalized Schouten bracket*. Its properties are the same as those of the usual Schouten bracket [44, 40, 22]. When the Lie algebroid is a tangent bundle $\tau_M : TM \to M$, that composition law reduces to the usual Schouten bracket . When the Lie algebroid is the cotangent bundle $\pi_P : T^*P \to P$ to a Poisson manifold (P, Π), the generalized Schouten bracket is the *bracket of forms* of all degrees on the Poisson manifold P, introduced by J.-L. Koszul [22], which extends the bracket of 1 forms used in Example 4 of 4.2.

3. Let $\varpi : A^* \to M$ be the dual bundle of the Lie algebroid $\pi : A \to M$. There exists on the space of sections of its exterior powers a graded endomorphism d_ρ, of degree 1 (this means that if η is a section of $\bigwedge^k A^*$, $d_\rho(\eta)$ is a section of $\bigwedge^{k+1} A^*$). That endomorphism satisfies

$$d_\rho \circ d_\rho = 0,$$

and its properties are essentially the same as those of the exterior derivative of differential forms. When the Lie algebroid is a tangent bundle $\tau_M : TM \to M$, d_ρ is the usual exterior derivative of differential forms.

We can develop on the spaces of sections of the exterior powers of a Lie algebroid and of its dual bundle a differential calculus very similar to the usual differential calculus of vector and multivector fields and differential forms on a manifold [6, 4, 34, 37]. Operators such as the interior product, the exterior derivative and the Lie derivative can still be defined and have properties similar to those of the corresponding operators for vector and multivector fields and differential forms on a manifold.

4.4 The Lie algebroid of a Lie groupoid

Let $\Gamma \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \Gamma_0$ be a Lie groupoid. Let $A(\Gamma)$ be the intersection of $\ker T\alpha$ and $T_{\Gamma_0}\Gamma$ (the tangent bundle $T\Gamma$ restricted to the submanifold Γ_0). We see that $A(\Gamma)$ is the total space of a vector bundle $\pi : A(\Gamma) \to \Gamma_0$, with base Γ_0, the canonical projection π being the map which associates a point $u \in \Gamma_0$ to every vector in $\ker T_u\alpha$. We will define a composition law on the set of smooth sections of that bundle, and a vector bundle map $\rho : A(\Gamma) \to T\Gamma_0$, for which $\pi : A(\Gamma) \to \Gamma_0$ is a Lie algebroid, callled the *Lie algebroid* of the Lie groupoid $\Gamma \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \Gamma_0$.

We observe first that for any point $u \in \Gamma_0$ and any point $x \in \beta^{-1}(u)$, the map $L_x : y \mapsto L_x y = m(x, y)$ is defined on the α-fibre $\alpha^{-1}(u)$, and maps that fibre onto the α-fibre $\alpha^{-1}\bigl(\alpha(x)\bigr)$. Therefore, $T_u L_x$ maps the vector space $A_u = \ker T_u\alpha$ onto the vector space $\ker T_x\alpha$, tangent at x to the α-fibre $\alpha^{-1}\bigl(\alpha(x)\bigr)$. Any vector $w \in A_u$ can therefore be extended into the vector field along $\beta^{-1}(u)$, $x \mapsto \widehat{w}(x) = T_u L_x(w)$. More generally, let $w : U \to A(\Gamma)$ be a smooth section of the vector bundle $\pi : A(\Gamma) \to \Gamma_0$, defined on an open subset U of Γ_0. By using the above described construction for every point $u \in U$, we can extend the section w into a smooth vector field $\widehat{w}$, defined on the open subset $\beta^{-1}(U)$ of Γ, by setting, for all $u \in U$ and $x \in \beta^{-1}(u)$,

$$\widehat{w}(x) = T_u L_x\bigl(w(u)\bigr).$$

We have defined an injective map $w \mapsto \widehat{w}$ from the space of smooth local sections of $\pi : A(\Gamma) \to \Gamma_0$, onto a subspace of the space of smooth vector fields defined on open subsets of Γ. The image of that map is the space of smooth vector fields $\widehat{w}$, defined on open subsets $\widehat{U}$ of Γ of the form $\widehat{U} = \beta^{-1}(U)$, where U is an open subset of Γ_0, which satisfy the two properties:

(i) $T\alpha \circ \widehat{w} = 0$,
(ii) for every x and $y \in \widehat{U}$ such that $\beta(x) = \alpha(y)$, $T_y L_x\big(\widehat{w}(y)\big) = \widehat{w}(xy)$.

These vector fields are called *left-invariant vector fields* on Γ.

One can easily see that the space of left-invariant vector fields on Γ is closed under the bracket operation. We can therefore define a composition law $(w_1, w_2) \mapsto \{w_1, w_2\}$ on the space of smooth sections of the bundle $\pi : A(\Gamma) \to \Gamma_0$ by defining $\{w_1, w_2\}$ as the unique section such that

$$\widehat{\{w_1, w_2\}} = [\widehat{w}_1, \widehat{w}_2].$$

Finally, we define the anchor map ρ as the map $T\beta$ restricted to $A(\Gamma)$. One can easily check that with that composition law and that anchor map, the vector bundle $\pi : A(\Gamma) \to \Gamma_0$ is a Lie algebroid, called the *Lie algebroid of* the Lie groupoid $\Gamma \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \Gamma_0$.

We could exchange the roles of α and β and use *right-invariant* vector fields instead of left-invariant vector fields. The Lie algebroid obtained remains the same, up to an isomorphism.

When the Lie groupoid $\Gamma \underset{\beta}{\overset{\alpha}{\rightrightarrows}}$ is a Lie group, its Lie algebroid is simply its Lie algebra.

4.5 The Lie algebroid of a symplectic groupoid

Let $\Gamma \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \Gamma_0$ be a symplectic groupoid, with symplectic form ω. As we have seen above, its Lie algebroid $\pi : A \to \Gamma_0$ is the vector bundle whose fibre, over each point $u \in \Gamma_0$, is $\ker T_u\alpha$. We define a linear map $\omega_u^\flat : \ker T_u\alpha \to T_u^*\Gamma_0$ by setting, for each $w \in \ker T_u\alpha$ and $v \in T_u\Gamma_0$,

$$\big\langle \omega_u^\flat(w), v \big\rangle = \omega_u(v, w).$$

Since $T_u\Gamma_0$ is Lagrangian and $\ker T_u\alpha$ complementary to $T_u\Gamma_0$ in the symplectic vector space $\big(T_u\Gamma, \omega(u)\big)$, the map $\omega_u^\flat$ is an isomorphism from $\ker T_u\alpha$ onto $T_u^*\Gamma_0$. By using that isomorphism for each $u \in \Gamma_0$, we obtain a vector bundle isomorphism of the Lie algebroid $\pi : A \to \Gamma_0$ onto the cotangent bundle $\pi_{\Gamma_0} : T^*\Gamma_0 \to \Gamma_0$.

As seen in Corollary 3.7, the submanifold of units Γ_0 has a unique Poisson structure Π for which $\alpha : \Gamma \to \Gamma_0$ is a Poisson map. Therefore, as seen in Example 4 of 4.2, the cotangent bundle $\pi_{\Gamma_0} : T^*\Gamma_0 \to \Gamma_0$ to the Poisson manifold (Γ_0, Π) has a Lie algebroid structure, with the bracket of 1-forms as composition law. That structure is the same as the structure obtained as a direct image of the Lie algebroid structure of $\pi : A(\Gamma) \to \Gamma_0$, by the above defined vector bundle isomorphism of $\pi : A \to \Gamma_0$ onto the cotangent bundle $\pi_{\Gamma_0} : T^*\Gamma_0 \to \Gamma_0$. The Lie algebroid of the symplectic groupoid $\Gamma \overset{\alpha}{\underset{\beta}{\rightrightarrows}} \Gamma_0$ can therefore be identified with the Lie algebroid $\pi_{\Gamma_0} : T^*\Gamma_0 \to \Gamma_0$, with its Lie algebroid structure of cotangent bundle to the Poisson manifold (Γ_0, Π).

4.6 The dual bundle of a Lie algebroid

There exist some very close relationships between Lie algebroids and Poisson manifolds, discussed, for example, in [11]. We have already seen one such relationship: the cotangent bundle of a Poisson manifold has a Lie algebroid structure. We now describe another one.

Let $\pi : A \to M$ be a Lie algebroid over a manifold M, with $\rho : A \to TM$ as anchor map. Let $\varpi : A^* \to M$ be the dual bundle of $\pi : A \to M$. We observe that a smooth section s of π can be considered as a smooth function on A^*, whose restriction to each fibre $\varpi^{-1}(x)$ $(x \in M)$ is the linear function $\zeta \mapsto \langle \zeta, s(x) \rangle$. By using that property, we see that the total space A^* has a unique Poisson structure, whose bracket extends, for smooth functions, the bracket of sections of π.

When the Lie algebroid is a finite-dimensional Lie algebra $\mathcal{G}$, the Poisson structure on its dual space $\mathcal{G}^*$ obtained by that means is the KKS-Poisson structure, already discussed in Section 1.

5 Integration of Lie algebroids and Poisson manifolds

According to Lie's third theorem, for any given finite-dimensional Lie algebra, there exists a Lie group whose Lie algebra is isomorphic to that Lie algebra. The same property is not true for Lie algebroids and Lie groupoids. The problem of finding necessary and sufficient conditions under which a given Lie algebroid is isomorphic to the Lie algebroid of a Lie groupoid remained open for more than 30 years. Partial results were obtained by J. Pradines [42], K. Mackenzie [33], P. Dazord [9], P. Dazord ans G. Hector [10]. A complete solution of that problem was obtained by M. Crainic and R. L. Fernandes [7]. Let us describe their very important work.

5.1 The Weinstein groupoid of a Lie algebroid

Admissible paths in a Lie algebroid

Let $\pi : A \to M$ be a Lie algebroid and $\rho : A \to TM$ its anchor map. Let $a : I = [0, 1] \to A$ be a smooth path in A. We will say that a is *admissible* if, for all $t \in I$,

$$\rho \circ a(t) = \frac{d}{dt}(\pi \circ a)(t).$$

We observe that when the Lie algebroid is the tangent bundle $\tau_M : TM \to M$, with the identity of TM as anchor map, a smooth path $a : I \to TM$ is admissible if and only if it is the canonical lift to TM of the smooth path $\tau_M \circ a : I \to M$ in M. For a general Lie algebroid, a smooth path $a : I \to A$ is admissible if and only if the map

$$\widehat{a} : TI = I \times \mathbb{R} \to A, \quad \widehat{a}(t, \lambda) = \lambda a(t),$$

is a Lie algebroid homomorphism of the Lie algebroid $p_1 : I \times \mathbb{R} \to I$ into the Lie algebroid $\pi : A \to M$, over the map $\pi \circ a : I \to M$ between the bases of these two Lie algebroids.

Associated paths in a Lie groupoid and in its Lie algebroid

Let us assume that $\pi : A \to M$ is the Lie algebroid of a Lie groupoid over M, $\Gamma \overset{\alpha}{\underset{\beta}{\rightrightarrows}} M$. Let $\gamma : I = [0,1] \to \Gamma$ be a smooth path in Γ, starting from a unit $x = \gamma(0) \in M \subset \Gamma$, and contained in the α-fibre $\alpha^{-1}(x)$, i.e., such that, for all $t \in I$,

$$\alpha \circ \gamma(t) = x.$$

For each $t \in I$, $\dfrac{d\gamma(t)}{dt}$ is a vector tangent to the fibre $\alpha^{-1}(x)$, at the point $\gamma(t)$. Applying to that vector a left translation by $\gamma(t)^{-1}$, we obtain a vector,

$$a_\gamma(t) = TL_{\gamma(t)^{-1}}\left(\frac{d\gamma(t)}{dt}\right) \in \ker T_{\beta\circ\gamma(t)}\alpha.$$

But we have seen (4.4) that for each $y \in M$, the vector subspace $\ker T_y\alpha$ of $T_y\Gamma$ is the fibre A_y over y of the Lie algebroid of the Lie groupoid $\Gamma \overset{\alpha}{\underset{\beta}{\rightrightarrows}} M$. The map $t \mapsto a_\gamma(t)$ is therefore a smooth path in the Lie algebroid $\pi : A \to M$. That path will be said to be *associated to* the path γ in the Lie groupoid $\Gamma \overset{\alpha}{\underset{\beta}{\rightrightarrows}} M$. Since we have

$$\beta \circ L_{\gamma(t)^{-1}} = \beta,$$

and since the anchor map of $\pi : A \to M$ is the restriction of $T\beta$, we see that the path a_γ is admissible.

Conversely, we see by integration that every smooth admissible path a in the Lie algebroid A is associated to a unique smooth path γ in the Lie groupoid Γ, starting from the unit $\gamma(0) = \pi \circ a(0) \in M$ and contained in the α-fibre $\alpha^{-1}\big(\gamma(0)\big)$.

A Lie groupoid with connected and simply connected α-fibres

We use the same assumptions as in the previous subsection. For smooth paths in Γ starting from a unit in M and contained in an α-fibre, homotopy with fixed endpoints is an equivalence relation. The one-to-one correspondence $\gamma \mapsto a_\gamma$, which associates, to each smooth path γ in Γ starting from a unit and contained in an α-fibre, a smooth admissible path a_γ in the Lie algebroid A, allows us to obtain an equivalence relation for smooth admissible paths in A. That equivalence relation will still be called *homotopy*.

Let $\gamma : I \to \Gamma$ and $\gamma' : I \to \Gamma$ be two smooth paths in Γ such that, for all $t \in I$,

$$\gamma(0) = \alpha \circ \gamma(t) = x \in M, \quad \gamma'(0) = \alpha \circ \gamma'(t) = y = \beta \circ \gamma(1) \in M.$$

We can define a path $\gamma.\gamma' : I \to \Gamma$ by setting

$$\gamma.\gamma'(t) = \begin{cases} \gamma(t) & \text{for } 0 \le t \le 1/2, \\ \gamma(1).\gamma'(2t-1) & \text{for } 1/2 \le t \le 1. \end{cases}$$

That path starts from the unit x and is contained in the α-fibre $\alpha^{-1}(x)$. By replacing γ and γ' by homotopic paths whose derivatives vanish on a neighbourhood of the endpoints 1 and 0, respectively, we may arrange things so that $\gamma.\gamma'$ is smooth (otherwise it is only piecewise smooth). By this means we obtain a composition law on the space of equivalence classes (for homotopy with fixed endpoints) of smooth paths in Γ, starting from a unit and contained in an α-fibre. Crainic and Fernandes have shown [7] that this space, endowed with that composition law, is a Lie groupoid with connected and simply connected α fibres, and that the Lie algebroid of that Lie groupoid is $\pi : A \to M$, the same as that of the Lie groupoid $\Gamma \underset{\beta}{\overset{\alpha}{\rightrightarrows}} M$. Moreover, they show that that Lie groupoid is, up to an isomorphism, the unique Lie groupoid with connected and simply connected α-fibres whose Lie algebroid is $\pi : A \to M$. That unicity property is the analogue of Lie's firt theorem for Lie algebras and Lie groups. It was also proved by Moerdijk and Mrčun [39] and by Mackenzie for transitive Lie algebroids [33].

Weinstein groupoid of a general Lie algebroid

We no longer assume now that the Lie algebroid $\pi : A \to M$ is the Lie algebroid of a Lie groupoid. We cannot use any more paths in the groupoid Γ, since Γ may not exist, but we still can use admissible paths in the Lie algebroid A. As seen above, when $\pi : A \to M$ is the Lie algebroid of the Lie groupoid $\Gamma \underset{\beta}{\overset{\alpha}{\rightrightarrows}} M$, homotopy with fixed endpoints for paths in Γ starting from a unit and contained in an α-fibre induces an equivalence relation, still called homotopy, on the space of admissible paths in A. That equivalence relation can be expressed in terms involving the Lie algebroid $\pi : A \to M$ only, and still makes sense when that Lie algebroid is no longer the Lie algebroid of a Lie groupoid. That key observation allows us to consider the quotient of the space of smooth admissible paths in A by the homotopy equivalence relation. That quotient will be denoted by $\mathcal{G}(A)$. Crainic and Fernandes have shown that $\mathcal{G}(A)$ is endowed with a natural topological groupoid structure, and have called it the *Weinstein* *groupoid* of the Lie algebroid $\pi : A \to M$ (the idea of that construction being due to A. Weinstein). When that Lie algebroid is the Lie algebroid of a Lie groupoid, its Weinstein groupoid $\mathcal{G}(A)$ is a Lie groupoid, and is in fact the unique Lie groupoid with connected and simply connected α-fibres whose Lie algebroid is $\pi : A \to M$.

Conversely, Crainic and Fernandes have shown that when the topological groupoid $\mathcal{G}(A)$ is in fact a Lie groupoid, its Lie algebroid is isomorphic to $\pi : A \to M$. The problem of finding necessary and sufficient conditions under which $\pi : A \to M$ is the Lie algebroid of a Lie groupoid amounts therefore to finding necessary and sufficient conditions under which the Weinstein groupoid $\mathcal{G}(A)$ is smooth, i.e., is a Lie groupoid. Crainic and Fernandes have solved that problem by introducing *monodromy groups*.

5.2 Monodromy groups

Let $\pi : A \to M$ be a Lie algebroid, $\rho : A \to TM$ its anchor map and $\mathcal{G}(A)$ its Weinstein groupoid. For each $x \in M$, let $A_x = \pi^{-1}(x)$, $\rho_x = \rho|_{A_x}$ and $\mathfrak{g}_x = \ker \rho_x$.

Let u and v be two elements in $\mathfrak{g}_x$, σ_u and σ_v two smooth sections of $\pi : A \to M$ such that $\sigma_u(x) = u$, $\sigma_v(x) = v$. The Leibniz rule shows that $\{\sigma_u, \sigma_v\}(x)$ depends only on u and v, not on the sections σ_u and σ_v. This allows us to define the bracket $[u, v]$ by setting

$$[u, v] = \{\sigma_u, \sigma_v\}(x).$$

With that bracket, $\mathfrak{g}_x$ is a Lie algebra.

Let $N_x(A)$ be the set of elements u in the center $Z(\mathfrak{g}_x)$ of the Lie algebra $\mathfrak{g}_x$ such that the constant path $t \mapsto u$, $t \in I = [0, 1]$, is equivalent (for the homotopy equivalence relation on the space of admissible smooth paths in A defined above) to the trivial path $t \mapsto 0_x$, where 0_x is the origin of the fibre A_x. Crainic and Fernandes have shown that N_x is a group, with the usual addition as composition law, which is at most countable, called the *monodromy group* at x. Let us take a Riemannian metric on the vector bundle $\pi : A \to M$, and let d be the associated distance in the fibres of A. We set

$$r(x) = \begin{cases} d\big(0_x, N_x(A) - \{0_x\}\big) & \text{if } N_x(A) - \{0_x\} \neq \emptyset, \\ +\infty & \text{if } N_x(A) - \{0_x\} = \emptyset. \end{cases}$$

We see that $r(x) > 0$ if and only if the monodromy group $N_x(A)$ is discrete. We may now state the main theorem.

Theorem 5.3 (Crainic and Fernandes [7]). *The Weinstein groupoid $\mathcal{G}(A)$ of the Lie algebroid $\pi : A \to M$ is a Lie groupoid if and only if the following two conditions are satisfied:*

(i) *for all $x \in M$, the monodromy group $N_x(A)$ is discrete, or in other words* $r(x) > 0$;

(ii) *for all $x \in M$, we have*

$$\liminf_{y \to x} r(y) > 0.$$

When these conditions are satisfied, the Lie algebroid $\pi : A \to M$ is said to be integrable.

The reader will find in [7] several examples of Lie algebroids for which the monodromy groups are fully determined. There are Lie algebroids which are locally integrable (it means that any point in the base space of the Lie algebroid has an open neighbourhood such that when restricted to that neighbourhood, the Lie algebroid is integrable), but not globally integrable; in other words, integrability of Lie algebroids is not a local property. There are also Lie algebroids which are not locally integrable.

5.4 Applications and comments

Integration of a Lie algebroid by a local Lie groupoid

One may think of a *local Lie groupoid* as the structure defined on a smooth manifold Γ by the following data: a smooth submanifold $\Gamma_0 \subset \Gamma$, smooth maps α, β, a composition law m and an inverse map ι, with the same properties as those of the corresponding maps of a Lie groupoid, but with the restriction that these maps are defined, and have these properties, only for elements in Γ close enough to the space Γ_0 of units. The Lie algebroid of a local Lie groupoid can be defined in the same way as that of a true Lie groupoid. J. Pradines [42] has shown that every Lie algebroid is the Lie algebroid of a *local* Lie groupoid. K. Mackenzie and P. Xu [35] have shown that when that Lie algebroid is the cotangent bundle to a Poisson manifold, the corresponding local Lie groupoid is a local symplectic groupoid. Crainic and Fernandes [7, 8] have shown that these results are now easy consequences of their general Theorem 5.3.

Some integrable Lie algebroids

By using Theorem 5.3, Crainic and Fernandes have proved the integrability of several types of Lie algebroids. Let us indicate some of their results

A Lie algebroid with zero anchor map (i.e., a sheaf of Lie algebras) is always integrable, and the corresponding Lie groupoid is a sheaf of Lie groups (that result was already found by Douady and Lazard [12]).

A regular Lie algebroid (i.e., a Lie algebroid whose anchor map is of constant rank) is locally integrable (but may be not globally integrable).

A Lie algebroid whose anchor map is injective, or more generally injective on a dense open set, is integrable.

5.5 Symplectic realizations of a Poisson manifold

The existence of local symplectic realizations of regular Poisson manifolds (i.e., with a Poisson structure of constant rank) was proved by Sophus Lie as early as 1890. The problem of finding a local symplectic realization of a Poisson manifold near a nonregular point is much more difficult. It was solved by Alan Weinstein in [48], where it is shown that if (P, Π) is a Poisson manifold of dimension n and $a \in P$ a point where the rank of Π is $2k$, there exists a symplectic realization $\varphi : M \to U$ of an open neighbourhood U of a by a symplectic manifold M of dimension $2(n-k)$. Moreover, that realization is universal, in the sense that if $\varphi' : M' \to U$ is another symplectic realization of U, then $\dim M' \geq \dim M$ and there exist (maybe after restriction of U and M') a surjective Poisson submersion $\chi : M' \to M$, whose fibres are symplectic submanifolds of M', such that $\varphi' = \varphi \circ \chi$.

By patching together symplectic realizations of open subsets of a Poisson manifold, A. Weinstein proved [49, 6] that any Poisson manifold has a global, surjective submersive symplectic realization. The same result was obtained independently by M. Karasev [18]. Another proof of that result was given by C. Albert and P. Dazord [1]. Weinstein and Karasev observed that such a realization automatically admits a local symplectic groupoid structure (this means a structure with a composition law, an inverse, source and target maps which look like those of a neighbourhood of the submanifold of units in a symplectic groupoid), the Poisson manifold being identified with the submanifold of units and the realization map either with the source or with the target map. The Lie algebroid of the (local) Lie groupoid obtained in that way can be identified with the cotangent bundle to the initially given Poisson manifold. The problem of finding a symplectic realization of a Poisson manifold (P, Π) appears therefore as closely linked to the problem of finding a Lie groupoid whose Lie algebroid is the cotangent bundle T^*P, with the Lie algebroid structure described in Example 4 of 4.2.

Integrable Poisson manifolds

A Poisson manifold (P, Π) is said to be *integrable* if there exists a symplectic groupoid $\Gamma \underset{\beta}{\overset{\alpha}{\rightrightarrows}} P$, with P as submanifolds of units, the Poisson structure Π being that for which α is a Poisson map and β an anti-Poisson map.

By using Theorem 5.3, Crainic and Fernandes have reinterpreted all known results about integrability of Poisson manifolds [8]. They have shown that a Poisson manifold (P, Π) is integrable if and only if its cotangent bundle T^*P, with the Lie algebroid structure described in Example 4 of 4.2, is an integrable Lie algebroid. The symplectic groupoid which integrates (P, Π) can be obtained as a symplectic quotient, as shown by Cattaneo and Felder [5]. They have shown that a Poisson manifold admits a complete realization if and only if it is integrable.

5.6 Examples

Let us indicate the symplectic groupoids which can be associated to some examples of Poisson manifolds.

1. Let $\mathcal{G}^*$ be the dual of a finite-dimensional Lie algebra $\mathcal{G}$, with its KKS-Poisson structure. Let G be a Lie group with Lie algebra $\mathcal{G}$. We can take as a symplectic groupoid associated to $\mathcal{G}^*$, the cotangent bundle T^*G with its natural Lie algebroid structure (described in 3.8). By choosing G connected and simply connected, we obtain a Lie groupoid structure on T^*G with connected and simply connected α- and β-fibres.

2. Let (M, ω) be a connected symplectic manifold. We can take as a symplectic groupoid associated to M (considered as a Poisson manifold) the product manifold $M \times M$, with the symplectic form $p_1^*\omega - p_2^*\omega$. We define on $M \times M$ the *pair groupoid* structure, for which the set of units is the diagonal $\Delta = \{(x, x); x \in M\}$, identified with M. The target and source maps are

$$\alpha(x, y) = (x, x), \qquad \beta(x, y) = (y, y), \quad x \text{ and } y \in M.$$

But there is a better choice for the symplectic groupoid associated to M: the *fundamental groupoid* of M, denoted by $\Pi(M)$. It is the set of homotopy classes (with fixed ends) of smooth paths $\varphi : [0, 1] \to M$. The set of units is now the set of homotopy classes of constant paths, which can be identified with M. The target and source maps associate, to the homotopy class $\widehat{\varphi}$ of a smooth path $\varphi : [0, 1] \to M$, the homotopy classes of the constant paths equal to $\varphi(0)$ and to $\varphi(1)$, respectively. We have natural projections $p_1 : \Pi(M) \to M$ and $p_2 : \Pi(M) \to M$, defined by $p_1(\widehat{\varphi}) = \varphi(0)$ and $p_2(\widehat{\varphi}) = \varphi(1)$. We take as a symplectic form on $\Pi(M)$ $p_1^*\omega - p_2^*\omega$. The fundamental groupoid $\Pi(M)$ has, over the pair groupoid $M \times M$, the advantage of having connected and simply connected α- and β-fibres.

In the next section we describe a third example: the symplectic groupoid associated to a Poisson–Lie group.

6 Double symplectic groupoid of a Poisson–Lie group

The notion of a Poisson–Lie group is due to Drinfel'd [13]. Let us recall its definition and some properties. The reader will find more details in [32, 50].

Definition 6.1. *A Poisson–Lie group is a Lie group G, equipped with a Poisson structure Π such that the product*

$$m : G \times G \to G, \quad (x, y) \mapsto m(x, y) = xy$$

is a Poisson map (when $G \times G$ is equipped with the natural product Poisson structure).

6.2 Properties and examples

1. Any Lie group can be considered as a Poisson–Lie group, with the zero Poisson structure.

2. The Poisson tensor Π of a Poisson–Lie group G vanishes at the unit element e of G. The linearized Poisson structure at that point is a skew-symmetric composition law on T_e^*G which satisfies the Jacobi identity. Therefore, the dual $\mathcal{G}^*$ of the Lie algebra $\mathcal{G}$ of G, which can be identified with T_e^*G, has a Lie algebra structure, determined by the Poisson structure on G. Therefore, we have a pair of vector spaces $(\mathcal{G}, \mathcal{G}^*)$, each of these spaces being the dual of the other, both equipped with a Lie algebra structure. The pair $(\mathcal{G}, \mathcal{G}^*)$ is called a *Lie bialgebra*.

3. The connected and simply connected Lie group G^* with Lie algebra $\mathcal{G}^*$ is called the *dual Lie group* of the Poisson–Lie group G. It also has a structure of Poisson–Lie group Π^*. When the Lie group G itself is connected and simply connected, the roles of G and G^* are the same, they can be exchanged (in other words, the dual Lie group of the Poisson–Lie group (G^*, Π^*) is the Poisson–Lie group (G, Π)).

4. On the vector space $\mathcal{D} = \mathcal{G} \oplus \mathcal{G}^*$, there is a natural Lie algebra structure, of which $\mathcal{G}$ and $\mathcal{G}^*$ are Lie subalgebras, whose bracket is

$$[X_1+\alpha_1, X_2+\alpha_2] = [X_1, X_2] - \operatorname{ad}^*_{\alpha_2} X_1 + \operatorname{ad}^*_{\alpha_1} X_2 + [\alpha_1, \alpha_2] + \operatorname{ad}^*_{X_1} \alpha_2 - \operatorname{ad}^*_{X_2} \alpha_1,$$

where X_1 and $X_2 \in \mathcal{G}$, α_1 and $\alpha_2 \in \mathcal{G}^*$. With that structure, $\mathcal{D}$ is called the *double Lie algebra* of $\mathcal{G}$.

5. The connected and simply connected Lie group D with Lie algebra $\mathcal{D}$ is called the *double Lie group* of the Poisson–Lie group G. When G is assumed to be connected and simply connected, there exist natural injective Lie group homomorphisms $G \to D$ and $G^* \to D$, which will be denoted by $g \mapsto \overline{g}$ and $u \mapsto \overline{u}$, respectively (with $g \in G$, $u \in G^*$, $\overline{g}$ and $\overline{u} \in D$).

6.3 Lu and Weinstein's construction

Let (G, Π) be a connected and simply connected Poisson–Lie group (the assumption of simple connexity is made for simplicity, and can easily be removed). J.-H. Lu and A. Weinstein [31] have obtained a very nice description of a symplectic groupoid whose set of units is (G, Π). We describe now that construction, with the above defined notations. Let

$$\Gamma = \left\{(g, u, v, h) \in G \times G^* \times G^* \times G; \overline{gu} = \overline{vh}\right\}.$$

Two different structures of Lie groupoid exist on Γ, the first one with G as a set of units and the second one with G^* as a set of units. We will denote by α_1, β_1, m_1 and ι_1 the target map, the source map, the composition law and the inverse map for the first structure, and by α_2, β_2, m_2 and ι_2 the corresponding maps for the second structure. They are given by the following formulae:

$$\alpha_1 : (g, u, v, h) \mapsto (g, e, e, g), \quad \beta_1 : (g, u, v, h) \mapsto (h, e, e, h),$$

$$m_1 : \big((g_1, u_1, v_1, h_1 = g_2), (g_2 = h_1, u_2, v_2, h_2) \mapsto (g_1, u_1u_2, v_1v_2, h_2),$$

$$\iota_1 : (g, u, v, h) \mapsto (h, u^{-1}, v^{-1}, g),$$

$$\alpha_2 : (g, u, v, h) \mapsto (e, v, v, e), \quad \beta_2 : (g, u, v, h) \mapsto (e, u, u, e),$$

$$m_2 : \big((g_1, u_1 = v_2, v_1, h_1), (g_2, u_2, v_2 = u_1, h_2) \mapsto (g_1g_2, u_2, v_1, h_1h_2),$$

$$\iota_2 : (g, u, v, h) \mapsto (g^{-1}, v, u, h^{-1}).$$

Moreover, Γ has a symplectic structure ω compatible with each of its two Lie groupoid structures; in other words, with that symplectic structure, it is a symplectic groupoid in two different ways, with either G or G^* as a set of units. Therefore, (Γ, ω) is a symplectic realization of each of the two Poisson–Lie groups (G, Π) and (G^*, Π^*).

7 Lie groupoid actions on a smooth manifold

We have seen that not every pair of elements in a Lie groupoid can be composed: a pair (x, y) can be composed if and only if the image $\beta(x)$, by the source map β, of the left element x, is equal to the image $\alpha(y)$, by the target map α, of the right element y. Similarly, to extend to a Lie groupoid action the well-known notion of a Lie group action on a smooth manifold, we must observe that not every pair made of an element in the groupoid and an element in the smooth manifold will be composable. For that reason, we need an extra ingredient, a map from the smooth manifold into the set of units of the goupoid, as shown in the following definition.

Definition 7.1. *Let* $\Gamma \overset{\alpha}{\underset{\beta}{\rightrightarrows}} \Gamma_0$ *be a Lie groupoid, M a smooth manifold and* $\mu : M \to \Gamma_0$ *a smooth map. Let*

$$\Gamma \times_\mu M = \big\{(x, m) \in \Gamma \times M; \beta(x) = \mu(m)\big\}.$$

We assume that $\Gamma \times_\mu M$ *is a smooth submanifold of* $\Gamma \times M$. (*This is true, for example, when μ is a submersion.*) *An* action on the left *of the Lie groupoid* $\Gamma \overset{\alpha}{\underset{\beta}{\rightrightarrows}} \Gamma_0$ *on the manifold M with* moment map μ *is a smooth map* $\Phi : \Gamma \times_\mu M \to M$ *which satisfies the following properties:*

(i) $\mu\big(\Phi(x, m)\big) = \alpha(x)$ *for all* $(x, m) \in \Gamma \times_\mu M$,
(ii) $\Phi\big(x, \Phi(y, m)\big) = \Phi(xy, m)$ *for all x and* $y \in \Gamma$, $m \in M$ *such that* $(x, y) \in \Gamma_2$ *and* $(y, m) \in \Gamma \times_\mu M$,
(iii) $\Phi\big(\mu(m), m\big) = m$ *for all* $m \in M$.

A similar definition holds for an action on the right of a Lie groupoid on a smooth manifold.

7.2 Properties and examples

1. A Lie groupoid $\Gamma \overset{\alpha}{\underset{\beta}{\rightrightarrows}} \Gamma_0$ acts on itself on the left with moment map α, and on on the right with moment map β.

2. A Lie groupoid $\Gamma \overset{\alpha}{\underset{\beta}{\rightrightarrows}} \Gamma_0$ acts on both sides on the submanifold of units Γ_0, with moment map id_{Γ_0}. These left and right actions are, respectively,

$$\big(x, u = \beta(x)\big) \mapsto \alpha(xu) = \alpha(x) \quad \text{and} \quad \big(u = \alpha(x), x\big) \mapsto \beta(ux) = \beta(x).$$

3. Property (ii) of the above definition implies that the moment map μ of a left action Φ of a Lie groupoid $\Gamma \overset{\alpha}{\underset{\beta}{\rightrightarrows}} \Gamma_0$ on a smooth manifold M satisfies, for all $x \in \Gamma$ and $m \in M$ such that $\beta(x) = \mu(m)$,

$$\mu\big(\Phi(x, m)\big) = \alpha\big(x\mu(m)\big) = \alpha(x).$$

In other words, the moment map μ is equivariant with respect to the left actions Φ of the Lie groupoid $\Gamma \overset{\alpha}{\underset{\beta}{\rightrightarrows}} \Gamma_0$ on M and the left action of that groupoid on its submanifold of units Γ_0. A similar property holds for right actions.

The following definition generalizes the notion of a symplectic (or Poisson) action of a Lie group on a symplectic (or Poisson) manifold.

Definition 7.3. *Let $\Gamma \overset{\alpha}{\underset{\beta}{\rightrightarrows}} \Gamma_0$ be a symplectic groupoid with symplectic form ω, (M, Π) a Poisson manifold and $\mu : M \to \Gamma_0$ a smooth map. Let $\Phi : \Gamma \times_\mu M \to M$ be a smooth action of the Lie groupoid $\Gamma \overset{\alpha}{\underset{\beta}{\rightrightarrows}} \Gamma_0$ on the manifold M, with moment map μ. That action is called a* Poisson action *if its graph*

$$\big\{(x, m, p) \in \Gamma \times M \times M;\, (x, m) \in \Gamma \times_\mu M,\ p = \Phi(x, m)\big\}$$

is a coisotropic submanifold of the product Poisson manifold $\Gamma \times M \times \overline{M}$, where Γ is equipped with the Poisson structure associated to its symplectic structure, M with the Poisson structure Π and where $\overline{M}$ means the manifold M equipped with the Poisson structure $-\Pi$.

K. Mikami and A. Weinstein [38] have developed a theory of reduction for Poisson actions of symplectic groupoids very similar to the classical reduction theory for symplectic actions.

The following proposition indicates an important property of the moment map for Poisson actions.

Proposition 7.4. *Let $\Phi : \Gamma \times_\mu M \to M$ be a Poisson action of the symplectic groupoid $\Gamma \overset{\alpha}{\underset{\beta}{\rightrightarrows}} \Gamma_0$ on the Poisson manifold (M, Π). The manifold of units Γ_0 will be equipped with the unique Poisson structure for which the target map $\alpha : \Gamma \to \Gamma_0$ is a Poisson map. Then the moment map $\mu : M \to \Gamma_0$ of that action is a Poisson map.*

Proof. Let s be a point in Γ_0, θ_1 and $\theta_2 \in T^*_s\Gamma_0$ and $n \in \mu^{-1}(s)$. Let $x \in \Gamma$, $m \in M$ be such that $\beta(x) = \mu(m)$ and that $\Phi(x, m) = n$.

Let ξ_1 and $\xi_2 \in T^*_x\Gamma$, η_1 and $\eta_2 \in T^*_m M$ and ζ_1 and $\zeta_2 \in T^*_n M$ be such that (ξ_1, η_1, ζ_1) and (ξ_2, η_2, ζ_2) belong to the annihilator of the tangent space at (x, m, n)

to the graph of Φ. Since that graph is a coisotropic submanifold of $\Gamma \times M \times \overline{M}$, we have

$$\Pi_\Gamma(\xi_1, \xi_2) + \Pi(\eta_1, \eta_2) - \Pi(\zeta_1, \zeta_2) = 0.$$

But by using the fact that $\mu \circ \Phi = \alpha \circ p_1$, where $p_1 : \Gamma \times_\mu M \to \Gamma$ is the projection on the first factor, we see that $(\alpha_x^* \theta_1, 0, -\mu_n^* \theta_1)$ and $(\alpha_x^* \theta_2, 0, -\mu_n^* \theta_2)$ belong to the annihilator of the tangent space at (x, m, n) to the graph of Φ. Therefore,

$$\Pi_\Gamma(\alpha_x^* \theta_1, \alpha_x^* \theta_2) - \Pi(\mu_n^* \theta_1, \mu_n^* \theta_2) = 0.$$

But since $\alpha : \Gamma \to \Gamma_0$ is a Poisson map,

$$\Pi_{\Gamma_0}(\theta_1, \theta_2) = \Pi_\Gamma(\alpha_x^* \theta_1, \alpha_x^* \theta_2) = \Pi(\mu_n^* \theta_1, \mu_n^* \theta_2).$$

That equality, which holds for all $s \in \Gamma_0$, θ_1 and $\theta_2 \in T_s^* \Gamma_0$ and $x \in \mu^{-1}(s)$, proves that μ is a Poisson map.

A shorter proof could be obtained by using the coisotropic calculus presented by A. Weinstein in [51], which extends to Poisson manifolds the calculus of symplectic relations initiated by W. M. Tulczyjew [46]. □

Conversely, P. Dazord [9] and P. Xu [55] have proved the following theorem.

Theorem 7.5. *Let $\Gamma \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \Gamma_0$ be a local symplectic groupoid and (M, Π) be a Poisson manifold. Every Poisson map $\mu : M \to \Gamma_0$ is the momentum map for a unique local Poisson action of $\Gamma \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \Gamma_0$ on (M, Π). If $\Gamma \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \Gamma_0$ is global (as a groupoid), α-connected and α-simply connected, that action is global if and only if μ is a complete Poisson map.*

Let us explain some terms used in the above statement. The symplectic groupoid $\Gamma \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \Gamma_0$ is said to be α-connected and α-simply connected if the α-fibres are connected and simply connected. The moment map μ is said to be a complete Poisson map if for every smooth function $f \in C^\infty(\Gamma_0, \mathbb{R})$ whose Hamiltonian vector field on Γ_0 is complete, the Hamiltonian vector field on M associated to the function $\mu^* f = f \circ \mu$ is complete.

7.6 Example

Let $\Phi : G \times M \to M$ be a Hamiltonian action (on the left) of a Lie group G on a Poisson manifold (M, Π), with an Ad^*-equivariant momentum map $J : M \to \mathcal{G}^*$. Let $T^*G \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \mathcal{G}^*$ be the cotangent space of G, equipped with its canonical symplectic groupoid structure (3.8). Let

$$T^*G \times_J M = \left\{(\xi, m) \in T^*G \times M;\, \beta(\xi) = J(m)\right\}.$$

We define an action $\widehat{\Phi} : T^*G \times_J M \to M$ of that symplectic groupoid on the Poisson manifold (M, Π), with moment map J, by setting, for all $(\xi, m) \in T^*G \times_J M$,

$$\widehat{\Phi}(\xi, m) = \Phi\bigl(\pi_G(\xi), m\bigr),$$

where $\pi_G : T^*M \to M$ is the canonical projection. That action is a Poisson action in the sense of Definition 7.3.

This example shows that Hamiltonian actions of Lie groups, with Ad^*-equivariant momentum maps, appear as special cases of Poisson actions of symplectic groupoids. By a modification of the symplectic structure on T^*G, that result can be extended to Hamiltonian actions whose momentum map is not Ad^*-equivariant, but rather equivariant with respect to an affine action of G on $\mathcal{G}^*$.

7.7 Poisson actions of Poisson–Lie groups

Let (G, Π_G) be a Poisson–Lie group and (P, Π_P) a Poisson manifold. An action on the left $\Phi : G \times P \to P$ of the Lie group G on the manifold P is called a *Poisson action* if its graph

$$\bigl\{(g, m, p) \in G \times M \times M;\ p = \Phi(g, m)\bigr\}$$

is a coisotropic submanifold of the product Poisson manifold $G \times M \times \overline{M}$, where $\overline{M}$ is the manifold M equipped with the Poisson structure $-\Pi$.

J.-H. Lu [30] has defined the notion of a momentum map for such an action. It is a map $J : P \to G^*$, with values in the dual Poisson–Lie group G^*, such that, for each X in the Lie algebra $\mathcal{G}$ of G,

$$X_P = \Pi_P^\sharp(J^* X^r),$$

where X^r is the right-invariant 1-form on G^* whose value at the unit element is $X \in \mathcal{G}$ (the vector space $\mathcal{G}$ being identified with the dual of $T_e G^*$, itself identified with $\mathcal{G}^*$). A momentum map in the sense of Lu is a Poisson map.

Along lines similar to those followed in 7.6, a Poisson–Lie group action on a Poisson manifold, with a momentum map in the sense of Lu, can be lifted to a Poisson action of the symplectic groupoid of G^*.

8 Poisson groupoids

Poisson groupoids were introduced by A. Weinstein [51] as a generalization of both symplectic groupoids and Poisson–Lie groups.

Definition 8.1. *A Poisson groupoid is a Lie groupoid* $\Gamma \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \Gamma_0$*, with a Poisson structure* Π *on* Γ*, such that the graph of the product*

$$\bigl\{(x, y, z) \in \Gamma \times \Gamma \times \Gamma;\ (x, y) \in \Gamma_2,\ z = xy\bigr\}$$

is a coisotropic submanifold of the product Poisson manifold $\Gamma \times \Gamma \times \overline{\Gamma}$*, where* $\overline{\Gamma}$ *is the manifold* Γ *with the Poisson structure* $-\Pi$.

8.2 Properties and examples

1. A symplectic groupoid is a Poisson groupoid whose Poisson structure is nondegenerate (i.e., symplectic).

2. A Poisson–Lie group is a Poisson groupoid whose groupoid structure is that of a Lie group (i.e., whose submanifold of units is reduced to one point).

3. As for a symplectic groupoid, the submanifold of units Γ_0 of a Poisson groupoid $\Gamma \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \Gamma_0$ has a unique Poisson structure for which the maps α and β are, respectively, a Poisson and a anti-Poisson map.

4. The Lie algebroid $\pi : A(\Gamma) \to \Gamma_0$ of a Poisson groupoid has an additional structure: its dual bundle $\varpi : A(\Gamma)^* \to \Gamma_0$ also has a Lie algebroid structure, compatible in a certain sense (indicated below) with that of $\pi : A(\Gamma) \to \Gamma_0$ (K. Mackenzie and P. Xu [34], Y. Kosmann-Schwarzbach [20], Z.-J. Liu and P. Xu [29]).

The compatibility condition between the two Lie algebroid structures on the two vector bundles in duality $\pi : A \to M$ and $\varpi : A^* \to M$ can be written as follows:

$$d_*[X, Y] = \mathcal{L}(X)d_*Y - \mathcal{L}(Y)d_*X,$$

where X and Y are two sections of π, or, using the generalized Schouten bracket (Property 2 of 4.3) of sections of exterior powers of the Lie algebroid $\pi : A \to M$,

$$d_*[X, Y] = [d_*X, Y] + [X, d_*Y].$$

In these formulae d_* is the generalized exterior derivative, which acts on the space of sections of exterior powers of the bundle $\pi : A \to M$, considered as the dual bundle of the Lie algebroid $\varpi : A^* \to M$, defined in Property 3 of 4.3.

These conditions are equivalent to the similar conditions obtained by exchange of the roles of A and A^*.

When the Poisson groupoid $\Gamma \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \Gamma_0$ is a symplectic groupoid, we have seen (4.5) that its Lie algebroid is the cotangent bundle $\pi_{\Gamma_0} : T^*\Gamma_0 \to \Gamma_0$ to the Poisson manifold Γ_0 (equipped with the Poisson structure for which α is a Poisson map). The dual bundle is the tangent bundle $\tau_{\Gamma_0} : T\Gamma_0 \to \Gamma_0$, with its natural Lie algebroid structure defined in Property 2 of 4.2.

5. Conversely, K. Mackenzie and P. Xu [35] have shown that if the Lie algebroid of a Lie groupoid is a Lie bialgebroid (that means, if there exists on the dual vector bundle of that Lie algebroid a compatible structure of Lie algebroid, in the above defined sense), that Lie groupoid has a Poisson structure for which it is a Poisson groupoid.

More information about Poisson groupoids can be found in [57].

9 Other developments

Poisson geometry, symplectic and Poisson groupoids and their generalizations, are currently very active fields of research, and several aspects could not be discussed here.

Let us quote the works of S. Bates and A. Weinstein [3] on geometric quantization, of J. Huebschmann [15] and P. Xu [56] on Poisson cohomology, of J.-C. Marrero and D. Iglesias [16, 17] on Lie groupoids associated to Jacobi structures [28], of M. Bangoura and Y. Kosmann-Schwarzbach [2, 20] on the dynamical Yang–Baxter equation and Gerstenhaber algebras, of P. Libermann [24, 25] on contact groupoids and Lie algebroids. We refer to the excellent review papers of Alan Weinstein [53, 54] for a much more extensive survey and other references.

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