

Differential calculus on a Lie algebroid and Poisson manifolds

Charles-Michel Marle

Université Pierre et Marie Curie, Institut de Mathématiques,
4, place Jussieu, 75252 Paris cedex 05, France

Abstract

A Lie algebroid over a manifold is a vector bundle over that manifold whose properties are very similar to those of a tangent bundle. Its dual bundle has properties very similar to those of a cotangent bundle: in the graded algebra of sections of its exterior powers, one can define an operator d_E similar to the exterior derivative. We present in this paper the theory of Lie derivatives, Schouten-Nijenhuis brackets and exterior derivatives in the general setting of a Lie algebroid, its dual bundle and their exterior powers. All the results (which, for their most part, are already known) are given with detailed proofs. In the final sections, the results are applied to Poisson manifolds.

1 Introduction

It is with great pleasure that I submit this work for a book in honour of my friend and colleague Professor José Antonio Pereira da Silva. I address my thanks to the organizers of that celebration for the opportunity to express my feelings of high esteem and friendship.

Lie groupoids and Lie algebroids were introduced in symplectic geometry by A. Weinstein [27, 4] and, independently, M. Karasev [9], in view of the symplectization of Poisson manifolds and applications to quantization. They are now

an active domain of research, with applications in various parts of mathematics [28, 3, 14]. More specifically, Lie algebroids have applications in Mechanics [15] and provide a very natural setting in which one can develop the theory of differential operators such as the exterior derivative of forms and the Lie derivative with respect to a vector field. In such a setting, slightly more general than that of the tangent and cotangent bundles to a smooth manifold and their exterior powers, the theory of Lie derivatives extends, in a very natural way, into the theory of the Schouten-Nijenhuis bracket (first introduced in differential geometry by J.A. Schouten [24] and developed by A. Nijenhuis [22]). Other bidifferential operators such as the bracket of exterior forms on a Poisson manifold, first discovered for Pfaff forms by F. Magri and C. Morosi [21] and extended to forms of all degrees by J.-L. Koszul [13] appear in such a setting as very natural: they are Schouten-Nijenhuis brackets for the Lie algebroid structure of the cotangent bundle to a Poisson manifold.

We present in this paper the theory of Lie derivatives, Schouten-Nijenhuis brackets and exterior derivatives in the general setting of a Lie algebroid, its dual bundle and their exterior powers. All the results (which, for their most part, are already known) are given with detailed proofs. Most of these proofs are the same as the classical ones (when the Lie algebroid is the tangent bundle to a smooth manifold); a few ones are slightly more complicated, because the algebra of sections of exterior powers of the dual of a Lie algebroid is not locally generated by its elements of degree 0 and their differentials (contrary to the algebra of exterior differential forms on a manifold). In the final section, the results are applied to Poisson manifolds.

2 Lie algebroids

The concept of a Lie algebroid was first introduced by J. Pradines [23], in relation with Lie groupoids.

2.1 Definition and examples

A Lie algebroid over a manifold is a vector bundle based on that manifold, whose properties are very similar to those of the tangent bundle. Let us give its formal definition.

2.1.1 Definition. Let M be a smooth manifold and (E, π, M) be a vector bundle with base M . A Lie algebroid structure on that bundle is the structure defined by the following data:

- a composition law $(s_1, s_2) \mapsto \{s_1, s_2\}$ on the space $\Gamma(\pi)$ of smooth sections of that bundle, for which $\Gamma(\pi)$ becomes a Lie algebra,
- a smooth vector bundle map $\rho : E \rightarrow TM$, where TM is the tangent bundle of M , which satisfies the following two properties:
 - (i) the map $s \mapsto \rho \circ s$ is a Lie algebras homomorphism from the Lie algebra $\Gamma(\pi)$ into the Lie algebra $A^1(M)$ of smooth vector fields on M ;
 - (ii) for every pair (s_1, s_2) of smooth sections of π , and every smooth function $f : M \rightarrow \mathbb{R}$, we have the Leibniz-type formula,

$$\{s_1, fs_2\} = f\{s_1, s_2\} + (i(\rho \circ s_1)df)s_2.$$

The vector bundle (E, π, M) equipped with its Lie algebroid structure will be called a Lie algebroid and denoted by (E, π, M, ρ) ; the composition law $(s_1, s_2) \mapsto \{s_1, s_2\}$ will be called the bracket and the map $\rho : E \rightarrow TM$ the anchor of the Lie algebroid (E, π, M, ρ) .

2.1.2 Remarks. Let (E, π, M, ρ) be a Lie algebroid.

- (i) The Lie algebras homomorphisms $s \mapsto \rho \circ s$ and $s \mapsto \mathcal{L}(\rho \circ s)$. Since $s \mapsto \rho \circ s$ is a Lie algebras homomorphism, we have, for every pair (s_1, s_2) of smooth sections of π ,

$$[\rho \circ s_1, \rho \circ s_2] = \rho \circ \{s_1, s_2\}.$$

For every smooth vector field X on M , let $\mathcal{L}(X)$ be the Lie derivative with respect to that vector field. We recall that $\mathcal{L}(X)$ is a derivation of $A = C^\infty(M, \mathbb{R})$, *i.e.*, that for every pair (f, g) of smooth functions on M ,

$$\mathcal{L}(X)(fg) = (\mathcal{L}(X)f)g + f(\mathcal{L}(X)g) .$$

We recall also that $X \mapsto \mathcal{L}(X)$ is a Lie algebras homomorphism from the Lie algebra $A^1(M)$ of smooth vector fields on M , into the Lie algebra $\text{Der}(C^\infty(M, \mathbb{R}))$ of derivations of $C^\infty(M, \mathbb{R})$, equipped with the commutator

$$(D_1, D_2) \mapsto [D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1$$

as composition law.

The map $s \mapsto \mathcal{L}(\rho \circ s)$, obtained by composition of two Lie algebras homomorphisms, is a Lie algebras homomorphism, from the Lie algebra $\Gamma(\pi)$ of smooth sections of the Lie algebroid (E, π, M, ρ) , into the Lie algebra of derivations of $C^\infty(M, \mathbb{R})$.

(ii) *Leibniz-type formulae.* According to Definition 2.1.1 we have, for any pair (s_1, s_2) of smooth sections of π and any smooth function f on M ,

$$\{s_1, fs_2\} = f\{s_1, s_2\} + (i(\rho \circ s_1)df) s_2 .$$

As an easy consequence of the definition, we also have

$$\{fs_1, s_2\} = f\{s_1, s_2\} - (i(\rho \circ s_2)df) s_1 .$$

More generally, for any pair (s_1, s_2) of smooth sections of π and any pair (f_1, f_2) of smooth functions on M , we have

$$\{f_1s_1, f_2s_2\} = f_1f_2\{s_1, s_2\} + f_1(i(\rho \circ s_1)df_2) s_2 - f_2(i(\rho \circ s_2)df_1) s_1 .$$

Using the Lie derivative operators, that formula may also be written as

$$\{f_1s_1, f_2s_2\} = f_1f_2\{s_1, s_2\} + f_1(\mathcal{L}(\rho \circ s_1)f_2) s_2 - f_2(\mathcal{L}(\rho \circ s_2)f_1) s_1 .$$

Let us give some examples of Lie algebroids.

2.1.3 Examples.

(i) *The tangent bundle.* The tangent bundle (TM, τ_M, M) of a smooth manifold M , equipped with the usual bracket of vector fields as composition law and with the identity map id_{TM} as anchor, is a Lie algebroid.

(ii) *An involutive distribution.* Let V be a smooth distribution on a smooth manifold M , *i.e.*, a smooth vector subbundle of the tangent bundle TM . We assume that V is involutive, *i.e.*, such that the space of its smooth sections is stable under the bracket operation. The vector bundle $(V, \tau_M|_V, M)$, with the usual bracket of vector fields as composition law and with the canonical injection $i_V : V \rightarrow TM$ as anchor, is a Lie algebroid. We have denoted by $\tau_M : TM \rightarrow M$ the canonical projection of the tangent bundle and by $\tau_M|_V$ its restriction to the subbundle V .

(iii) *A sheaf of Lie algebras.* Let (E, π, M) be a vector bundle over the smooth manifold M and $(z_1, z_2) \mapsto [z_1, z_2]$ be a smooth, skew-symmetric bilinear bundle map defined on the fibered product $E \times_M E$, with values in E , such that for each $x \in M$, the fibre $E_x = \pi^{-1}(x)$, equipped with the bracket $(z_1, z_2) \mapsto [z_1, z_2]$, is a Lie algebra. We define the bracket of two smooth sections s_1 and s_2 of π as the section $\{s_1, s_2\}$ such that, for each $x \in M$, $\{s_1, s_2\}(x) = [s_1(x), s_2(x)]$. For the anchor, we take the zero vector bundle map from E to TM . Then (E, π, M) is a Lie algebroid of particular type, called a *sheaf of Lie algebras* over the manifold M .

(iv) *A finite-dimensional Lie algebra.* In particular, a finite-dimensional Lie algebra can be considered as a Lie algebroid over a base reduced to a single point, with the zero map as anchor.

(v) *The Lie algebroid of a Lie groupoid.* To every Lie groupoid, there is an associated Lie algebroid, much like to every Lie group there is an associated Lie algebra. It is in that setting that Pradines [23] introduced Lie algebroids for the first time. For more informations about Lie groupoids and their associated Lie algebroids, the reader is referred to [19, 4, 5, 1].

2.2 Locality of the bracket

We will prove that the value, at any point $x \in M$, of the bracket of two smooth sections s_1 and s_2 of the Lie algebroid (E, π, M, ρ) , depends only on the jets of order 1 of s_1 and s_2 at x . We will need the following lemma.

2.2.1 Lemma. *Let (E, π, M, ρ) be a Lie algebroid, $s_1 : M \rightarrow E$ a smooth section of π , and U an open subset of M on which s_1 vanishes. Then for any other smooth section s_2 of π , $\{s_1, s_2\}$ vanishes on U .*

Proof:

Let x be a point in U . There exists a smooth function $f : M \rightarrow \mathbb{R}$, whose support is contained in U such that $f(x) = 1$. The section fs_1 vanishes identically, since s_1 vanishes on U while f vanishes outside of U . Therefore, for any other smooth section s_2 of π ,

$$0 = \{fs_1, s_2\} = -\{s_2, fs_1\} = -f\{s_2, s_1\} - (i(\rho \circ s_2)df)s_1.$$

So at x we have

$$f(x)\{s_1, s_2\}(x) = (i(\rho \circ s_2)df)(x)s_1(x) = 0.$$

Since $f(x) = 1$, we obtain $\{s_1, s_2\}(x) = 0$. \(\boxtimes\)

2.2.2 Proposition. *Let (E, π, M, ρ) be a Lie algebroid. The value $\{s, s'\}(x)$ of the bracket of two smooth sections s and s' of π , at a point $x \in M$, depends only on the jets of order 1 of s and s' at x . Moreover, if $s(x) = 0$ and $s'(x) = 0$, then $\{s, s'\}(x) = 0$.*

Proof:

Let U be an open neighbourhood of x in M on which there exists a local basis $(\sigma_1, \dots, \sigma_k)$ of smooth sections of π . For any point $y \in U$, $(\sigma_1(y), \dots, \sigma_k(y))$ is a basis of the fibre $E_y = \pi^{-1}(y)$. Let s_1 and s_2 be two smooth sections of π . On the open subset U , these two sections can be expressed, in a unique way, as

$$s_1 = \sum_{i=1}^k f_i \sigma_i, \quad s_2 = \sum_{j=1}^k g_j \sigma_j,$$

where the f_i and g_j are smooth functions on U .

By Lemma 2.2.1, the values of $\{s_1, s_2\}$ in U depend only on the values of s_1 and s_2 in U . Therefore we have in U

$$\{s_1, s_2\} = \sum_{i,j} \left(f_i g_j \{\sigma_i, \sigma_j\} + f_i (\mathcal{L}(\rho \circ \sigma_i) g_j) \sigma_j - g_j (\mathcal{L}(\rho \circ \sigma_j) f_i) \sigma_i \right).$$

This expression proves that the value of $\{s_1, s_2\}$ at x depends only on the $f_i(x)$, $df_i(x)$, $g_j(x)$ and $dg_j(x)$, that means on the jets of order 1 of s_1 and s_2 at x .

If $s_1(x) = 0$, we have, for all $i \in \{1, \dots, k\}$, $f_i(x) = 0$, and similarly if $s_2(x) = 0$, we have, for all $j \in \{1, \dots, k\}$, $g_j(x) = 0$. The above expression shows then that $\{s_1, s_2\}(x) = 0$. \bowtie

3 Exterior powers of vector bundles

We recall in this section some definitions and general properties related to vector bundles, their dual bundles and exterior powers. In a first subsection we recall some properties of graded algebras, graded Lie algebras and their derivations. The second subsection applies these properties to the graded algebra of sections of the exterior powers of a vector bundle. For more details the reader may look at the book by Greub, Halperin and Vanstone [6]. The reader already familiar with this material may skip this section, or just look briefly at the sign conventions we are using.

3.1 Graded vector spaces and graded algebras

3.1.1 Definitions.

(i) An algebra is a vector space A on the field $\mathbb{K} = \mathbb{R}$ or \mathbb{C} , endowed with a \mathbb{K} -bilinear map called the composition law,

$$A \times A \rightarrow A, \quad (x, y) \mapsto xy, \quad \text{where } x \text{ and } y \in A.$$

(ii) An algebra A is said to be *associative* if its composition law is associative, i.e., if for all x, y and $z \in A$,

$$x(yz) = (xy)z.$$

(iii) A vector space E on the field $\mathbb{K} = \mathbb{R}$ or \mathbb{C} is said to be \mathbb{Z} -graded if one has chosen a family $(E^p, p \in \mathbb{Z})$ of vector subspaces of E , such that

$$E = \bigoplus_{p \in \mathbb{Z}} E^p.$$

For each $p \in \mathbb{Z}$, an element $x \in E$ is said to be *homogeneous of degree p* if $x \in E^p$, and the vector subspace E^p of E is called the *subspace of elements homogeneous of degree p* .

(iv) Let $E = \bigoplus_{p \in \mathbb{Z}} E^p$ and $F = \bigoplus_{p \in \mathbb{Z}} F^p$ be two \mathbb{Z} -graded vector spaces on the same field \mathbb{K} . A \mathbb{K} -linear map $f : E \rightarrow F$ is said to be *homogeneous of degree d* (with $d \in \mathbb{Z}$) if for each $p \in \mathbb{Z}$,

$$f(E^p) \subset F^{p+d}.$$

(v) An algebra A is said to be \mathbb{Z} -graded if $A = \bigoplus_{p \in \mathbb{Z}} A^p$ is \mathbb{Z} -graded as a vector space and if in addition, for all p and $q \in \mathbb{Z}$, $x \in A^p$ and $y \in A^q$,

$$xy \in A^{p+q}.$$

(vi) A \mathbb{Z} -graded algebra $A = \bigoplus_{p \in \mathbb{Z}} A^p$ is said to be \mathbb{Z}_2 -commutative (resp., \mathbb{Z}_2 -anticommutative) if for all p and $q \in \mathbb{Z}$, $x \in A^p$ and $y \in A^q$,

$$xy = (-1)^{pq}yx, \quad (\text{resp.,} \quad xy = -(-1)^{pq}yx).$$

3.1.2 Some properties and examples

(i) *Composition of homogeneous linear maps.* We consider three \mathbb{Z} -graded vector spaces, $E = \bigoplus_{p \in \mathbb{Z}} E^p$, $F = \bigoplus_{p \in \mathbb{Z}} F^p$ and $G = \bigoplus_{p \in \mathbb{Z}} G^p$, on the same field \mathbb{K} . Let $f : E \rightarrow F$ and $g : F \rightarrow G$ be two linear maps, the first one f being

homogeneous of degree d_1 and the second one g homogeneous of degree d_2 . Then $g \circ f : E \rightarrow G$ is homogeneous of degree $d_1 + d_2$.

(ii) *The algebra of linear endomorphisms of a vector space.* Let E be a vector space and $\mathcal{L}(E, E)$ be the space of linear endomorphisms of E . We take as composition law on that space the usual composition of maps,

$$(f, g) \mapsto f \circ g, \quad \text{with} \quad f \circ g(x) = f(g(x)), \quad x \in E.$$

With that composition law, $\mathcal{L}(E, E)$ is an associative algebra.

(iii) *The graded algebra of graded linear endomorphisms.* We assume now that $E = \bigoplus_{p \in \mathbb{Z}} E^p$ is a \mathbb{Z} -graded vector space. For each $d \in \mathbb{Z}$, let A^d be the vector subspace of $\mathcal{L}(E, E)$ whose elements are the linear endomorphisms $f : E \rightarrow E$ which are homogeneous of degree d , i.e., such that for all $p \in \mathbb{Z}$, $f(E^p) \subset E^{p+d}$. Let $A = \bigoplus_{d \in \mathbb{Z}} A^d$. By using property 3.1.2 (i), we see that with the usual composition of maps as composition law, A is a \mathbb{Z} -graded associative algebra.

Let us use property 3.1.2 (i) with $E = F = G$, in the following definition.

3.1.3 Definition. Let $E = \bigoplus_{p \in \mathbb{Z}} E^p$ be a \mathbb{Z} -graded vector space, f and $g \in \mathcal{L}(E, E)$ be two homogeneous linear endomorphisms of E of degrees d_1 and d_2 , respectively. The linear endomorphism $[f, g]$ of E defined by

$$[f, g] = f \circ g - (-1)^{d_1 d_2} g \circ f,$$

which, by 3.1.2 (i), is homogeneous of degree $d_1 + d_2$, is called the *graded bracket* of f and g .

3.1.4 Definition. Let $A = \bigoplus_{p \in \mathbb{Z}} A^p$ be a \mathbb{Z} -graded algebra. Let $\theta : A \rightarrow A$ be a linear endomorphism of the graded vector space A . Let $d \in \mathbb{Z}$. The linear endomorphism θ is said to be a *derivation of degree d* of the graded algebra A if

- (i) as a linear endomorphism of a graded vector space, θ is homogeneous of degree d ,

(ii) for all $p \in \mathbb{Z}$, $x \in A^p$ and $y \in A$,

$$\theta(xy) = (\theta(x))y + (-1)^{d_p}x(\theta(y)) .$$

3.1.5 Remark. More generally, as shown by Koszul [12], for an algebra A equipped with an involutive automorphism, one can define two types of remarkable linear endomorphisms of A , the *derivations* and the *antiderivations*. When $A = \bigoplus_{p \in \mathbb{Z}} A^p$ is a \mathbb{Z} -graded algebra, and when the involutive automorphism used is that which maps each $x \in A^p$ onto $(-1)^p x$, it turns out that all nonzero graded derivations are of even degree, that all nonzero graded antiderivations are of odd degree, and that both derivations and antiderivations can be defined as done in Definition 3.1.4. For simplicity we have chosen to call *derivations* both the derivations and antiderivations.

3.1.6 Some properties of derivations

Let $A = \bigoplus_{p \in \mathbb{Z}} A^p$ be a \mathbb{Z} -graded algebra.

(i) *A derivation of degree 0.* For every $p \in \mathbb{Z}$ and $x \in A^p$, we set

$$\mu(x) = px .$$

The map μ , defined for homogeneous elements of A , can be extended in a unique way as a linear endomorphism of A , still denoted by μ . This endomorphism is a derivation of degree 0 of A .

(ii) *The graded bracket of two derivations.* Let $\theta_1 : A \rightarrow A$ and $\theta_2 : A \rightarrow A$ be two derivations of A , of degree d_1 and d_2 , respectively. Their graded bracket (Definition 3.1.3)

$$[\theta_1, \theta_2] = \theta_1 \circ \theta_2 - (-1)^{d_1 d_2} \theta_2 \theta_1 ,$$

is a derivation of degree $d_1 + d_2$.

3.1.7 Definition. A \mathbb{Z} -graded Lie algebra is a \mathbb{Z} -graded algebra $A = \bigoplus_{p \in \mathbb{Z}} A^p$ (in the sense of 3.1.1 (v)), whose composition law, often denoted by $(x, y) \mapsto [x, y]$ and called the *graded bracket*, satisfies the following two properties:

- (i) it is \mathbb{Z}_2 -anticommutative in the sense of 3.1.1 (vi), i.e., for all p and $q \in \mathbb{Z}$, $P \in A^p$ and $Q \in A^q$,

$$[P, Q] = -(-1)^{pq}[Q, P],$$

- (ii) it satisfies the \mathbb{Z} -graded Jacobi identity, i.e., for p, q and $r \in \mathbb{Z}$, $P \in A^p$, $Q \in A^q$ and $R \in A^r$,

$$(-1)^{pr}[P, [Q, R]] + (-1)^{qp}[Q, [R, P]] + (-1)^{rq}[R, [P, Q]] = 0.$$

3.1.8 Examples and remarks

(i) *Lie algebras and \mathbb{Z} -graded Lie algebras.* A \mathbb{Z} -graded Lie algebra $A = \bigoplus_{p \in \mathbb{Z}} A^p$ is not a Lie algebra in the usual sense, unless $A^p = \{0\}$ for all $p \neq 0$. However, its subspace A^0 of homogeneous elements of degree 0 is a Lie algebra in that usual sense: it is stable under the bracket operation and when restricted to elements in A^0 , the bracket is skew-symmetric and satisfies the usual Jacobi identity.

(ii) *The graded Lie algebra associated to a graded associative algebra.* Let $A = \bigoplus_{p \in \mathbb{Z}} A^p$ be a \mathbb{Z} -graded associative algebra, whose composition law is denoted by $(P, Q) \mapsto PQ$. We define another composition law, denoted by $(P, Q) \mapsto [P, Q]$ and called the *graded commutator*; we first define it for homogeneous elements in A by setting, for all p and $q \in \mathbb{Z}$, $P \in A^p$ and $Q \in A^q$,

$$[P, Q] = PQ - (-1)^{pq}QP;$$

then we extend the definition of that composition law to all pairs of elements in A by bilinearity. The reader will easily verify that with this composition law, A is a graded Lie algebra. When $A^p = \{0\}$ for all $p \neq 0$, we recover the well known way in which one can associate a Lie algebra to any associative algebra.

(iii) *The graded Lie algebra of graded endomorphisms.* Let $E = \bigoplus_{p \in \mathbb{Z}} E^p$ be a graded vector space. For each $p \in \mathbb{Z}$, let $A^p \subset \mathcal{L}(E, E)$ be the space of linear endomorphisms of E which are homogeneous of degree p , and let $A =$

$\bigoplus_{p \in \mathbb{Z}} A^p$. As we have seen in 3.1.2 (iii), when equipped with the composition of applications as composition law, A is a \mathbb{Z} -graded associative algebra. Let us define another composition law on A , called the *graded commutator*; we first define it for homogeneous elements in A by setting, for all p and $q \in \mathbb{Z}$, $P \in A^p$ and $Q \in A^q$,

$$[P, Q] = PQ - (-1)^{pq}QP;$$

then we extend the definition of that composition law to all pairs of elements in A by bilinearity. By using 3.1.8 (ii), we see that A , with this composition law, is a \mathbb{Z} -graded Lie algebra.

(iv) *Various interpretations of the graded Jacobi identity.* Let $A = \bigoplus_{p \in \mathbb{Z}} A^p$ be a \mathbb{Z} -graded Lie algebra. The \mathbb{Z} -graded Jacobi identity indicated in Definition 3.1.7 can be cast into other forms, which better indicate its meaning. Let us set, for all P and $Q \in A$,

$$\text{ad}_P Q = [P, Q].$$

For each $p \in \mathbb{Z}$ and $P \in A^p$, $\text{ad}_P : A \rightarrow A$ is a graded endomorphism of A , homogeneous of degree p . By taking into account the \mathbb{Z}_2 -anticommutativity of the bracket, the reader will easily see that the graded Jacobi identity can be written under the following two forms:

First form. For all p, q and $r \in \mathbb{Z}$, $P \in A^p$, $Q \in A^q$ and $R \in A^r$,

$$\text{ad}_P([Q, R]) = [\text{ad}_P Q, R] + (-1)^{pq}[Q, \text{ad}_P R].$$

This equality means that for all $p \in \mathbb{Z}$ and $P \in A^p$, the linear endomorphism $\text{ad}_P : A \rightarrow A$ is a derivation of degree p of the graded Lie algebra A , in the sense of 3.1.4.

Second form. For all p, q and $r \in \mathbb{Z}$, $P \in A^p$, $Q \in A^q$ and $R \in A^r$,

$$\text{ad}_{[P, Q]} R = \text{ad}_P \circ \text{ad}_Q R - (-1)^{pq} \text{ad}_Q \circ \text{ad}_P R = [\text{ad}_P, \text{ad}_Q] R.$$

This equality means that for all p and $q \in \mathbb{Z}$, $P \in A^p$ and $Q \in A^q$, the endomorphism $\text{ad}_{[P, Q]} : A \rightarrow A$ is the graded bracket (in the sense of 3.1.3) of

the two endomorphisms $\text{ad}_P : A \rightarrow A$ and $\text{ad}_Q : A \rightarrow A$. In other words, the map $P \mapsto \text{ad}_P$ is a \mathbb{Z} -graded Lie algebras homomorphism from the \mathbb{Z} -graded Lie algebra A into the \mathbb{Z} -graded Lie algebra of sums of linear homogeneous endomorphisms of A , with the graded bracket as composition law (example 3.1.8 (iii)).

When $A^p = \{0\}$ for all $p \neq 0$, we recover the well known interpretations of the usual Jacobi identity.

3.2 Exterior powers of a vector bundle and of its dual

In what follows all the vector bundles will be assumed to be locally trivial and of finite rank; therefore we will write simply *vector bundle* for *locally trivial vector bundle*.

3.2.1 The dual of a vector bundle

Let (E, π, M) be a vector bundle on the field $\mathbb{K} = \mathbb{R}$ or \mathbb{C} . We will denote its *dual bundle* by (E^*, ϖ, M) . Let us recall that it is a vector bundle over the same base manifold M , whose fibre $E_x^* = \varpi^{-1}(x)$ over each point $x \in M$ is the dual vector space of the corresponding fibre $E_x = \pi^{-1}(x)$ of (E, π, M) , *i.e.*, the space of linear forms on E_x (*i.e.*, linear functions defined on E_x and taking their values in the field \mathbb{K}).

For each $x \in M$, the duality coupling $E_x^* \times E_x \rightarrow \mathbb{K}$ will be denoted by

$$(\eta, v) \mapsto \langle \eta, v \rangle.$$

3.2.2 The exterior powers of a vector bundle

Let (E, π, M) be a vector bundle of rank k . For each integer $p > 0$, we will denote by $(\wedge^p E, \pi, M)$ the p -th external power of (E, π, M) . It is a vector bundle over M whose fibre $\wedge^p E_x$, over each point $x \in M$, is the p -th external power of the corresponding fibre $E_x = \pi^{-1}(x)$ of (E, π, M) . We recall that $\wedge^p E_x$ can be canonically identified with the vector space of p -multilinear skew-symmetric forms on the dual E_x^* of E_x .

Similarly, for any integer $p > 0$, we will denote by $(\wedge^p E^*, \varpi, M)$ the p -th external power of the bundle (E^*, ϖ, M) , dual of (E, π, M) .

For $p = 1$, $(\wedge^1 E, \pi, M)$ is simply the bundle (E, π, M) , and similarly $(\wedge^1 E^*, \varpi, M)$ is simply the bundle (E^*, ϖ, M) . For p strictly larger than the rank k of (E, π, M) , $(\wedge^p E, \pi, M)$ and $(\wedge^p E^*, \varpi, M)$ are the trivial bundle over M , $(M \times \{0\}, \pi_1, M)$, whose fibres are zero-dimensional ($\pi_1 : M \times \{0\} \rightarrow M$ being the projection onto the first factor).

For $p = 0$, we will set $(\wedge^0 E, \pi, M) = (\wedge^0 E^*, \varpi, M) = (M \times \mathbb{K}, \pi_1, M)$, where $\pi_1 : M \times \mathbb{K} \rightarrow M$ is the projection onto the first factor.

Finally, we will consider that for $p < 0$, $(\wedge^p E, \pi, M)$ and $(\wedge^p E^*, \varpi, M)$ are the trivial bundle over M , $(M \times \{0\}, \pi_1, M)$. With these conventions, $(\wedge^p E, \pi, M)$ and $(\wedge^p E^*, \varpi, M)$ are defined for all $p \in \mathbb{Z}$.

3.2.3 Operations in the graded vector spaces $\wedge E_x$ and $\wedge E_x^*$

Let (E, π, M) be a vector bundle of rank k , (E^*, ϖ, M) its dual and, for each $p \in \mathbb{Z}$, $(\wedge^p E, \pi, M)$ and $(\wedge^p E^*, \varpi, M)$ their p -th external powers. We recall in this sections some operations which can be made for each point $x \in M$, in the vector spaces $\wedge^p E_x$ and $\wedge^p E_x^*$.

For each $x \in M$, let us consider the \mathbb{Z} -graded vector spaces

$$\wedge E_x = \bigoplus_{p \in \mathbb{Z}} \wedge^p E_x \quad \text{and} \quad \wedge E_x^* = \bigoplus_{p \in \mathbb{Z}} \wedge^p E_x^*.$$

We will say that elements in $\wedge E_x^*$ are (multilinear) *forms* at x , and that elements in $\wedge E_x$ are *multivectors* at x .

(i) *The exterior product.* Let us recall that for each $x \in M$, p and $q \in \mathbb{Z}$, $P \in \wedge^p E_x$ and $Q \in \wedge^q E_x$, there exists $P \wedge Q \in \wedge^{p+q} E_x$, called the *exterior product* of P and Q , defined by the following formulae.

- If $p < 0$, then $P = 0$, therefore, for any $Q \in \wedge^q E_x$, $P \wedge Q = 0$. Similarly, if $q < 0$, then $Q = 0$, therefore, for any $P \in \wedge^p E_x$, $P \wedge Q = 0$.

- If $p = 0$, then P is a scalar ($P \in \mathbb{K}$), and therefore, for any $Q \in \bigwedge^q E_x$, $P \wedge Q = PQ$, the usual product of Q by the scalar P . Similarly, for $q = 0$, then Q is a scalar ($Q \in \mathbb{K}$), and therefore, for any $P \in \bigwedge^p E_x$, $P \wedge Q = QP$, the usual product of P by the scalar Q .
- If $p \geq 1$ and $q \geq 1$, $P \wedge Q$, considered as a $(p+q)$ -multilinear form on E_x^* , is given by the formula, where $\eta_1, \dots, \eta_{p+q} \in E_x^*$,

$$P \wedge Q(\eta_1, \dots, \eta_{p+q}) = \sum_{\sigma \in \mathfrak{S}_{(p,q)}} \varepsilon(\sigma) P(\eta_{\sigma(1)}, \dots, \eta_{\sigma(p)}) Q(\eta_{\sigma(p+1)}, \dots, \eta_{\sigma(p+q)}).$$

We have denoted by $\mathfrak{S}_{(p,q)}$ the set of permutations σ of $\{1, \dots, p+q\}$ which satisfy

$$\sigma(1) < \sigma(2) < \dots < \sigma(p) \quad \text{and} \quad \sigma(p+1) < \sigma(p+2) < \dots < \sigma(p+q),$$

and set

$$\epsilon(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ is even,} \\ -1 & \text{if } \sigma \text{ is odd.} \end{cases}$$

Similarly, let us recall that for each $x \in M$, p and $q \in \mathbb{Z}$, $\xi \in \bigwedge^p E_x^*$ and $\eta \in \bigwedge^q E_x^*$, there exists $\xi \wedge \eta \in \bigwedge^{p+q} E_x^*$, called the *exterior product* of ξ and η . It is defined by the formulae given above, the only change being the exchange of the roles of E_x and E_x^* .

The exterior product is associative and \mathbb{Z}_2 -commutative: for all $x \in M$, p, q and $r \in \mathbb{Z}$, $P \in \bigwedge^p E_x$, $Q \in \bigwedge^q E_x$ and $R \in \bigwedge^r E_x$,

$$P \wedge (Q \wedge R) = (P \wedge Q) \wedge R, \quad Q \wedge P = (-1)^{pq} P \wedge Q,$$

and similarly, for $\xi \in \bigwedge^p E_x^*$, $\eta \in \bigwedge^q E_x^*$ and $\zeta \in \bigwedge^r E_x^*$,

$$\xi \wedge (\eta \wedge \zeta) = (\xi \wedge \eta) \wedge \zeta, \quad \eta \wedge \xi = (-1)^{pq} \xi \wedge \eta.$$

For all $x \in M$, the exterior product extends, by bilinearity, as a composition law in each of the graded vector spaces $\bigwedge E_x$ and $\bigwedge E_x^*$. With these composition laws, these vector spaces become \mathbb{Z} -graded associative and \mathbb{Z}_2 -commutative algebras.

(ii) *The interior product of a form by a vector.* Let us recall that for each $x \in M$, $v \in E_x$, $p \in \mathbb{Z}$, $\eta \in \bigwedge^p E_x^*$, there exists $i(v)\eta \in \bigwedge^{p-1} E_x^*$, called the *interior product* of η by v , defined by the following formulae.

– For $p \leq 0$, $i(v)\eta = 0$, since $\bigwedge^{p-1} E_x^* = \{0\}$.

– For $p = 1$,

$$i(v)\eta = \langle \eta, v \rangle \in \mathbb{K}.$$

– For $p > 1$, $i(v)\eta$ is the $(p-1)$ -multilinear form on E_x such that, for all $v_1, \dots, v_{p-1} \in E_x$,

$$i(v)\eta(v_1, \dots, v_{p-1}) = \eta(v, v_1, \dots, v_{p-1}).$$

For each $x \in M$ and $v \in E_x$, the map $\eta \mapsto i(v)\eta$ extends, by linearity, as a graded endomorphism of degree -1 of the graded vector space $\bigwedge E_x^*$. Moreover, that endomorphism is in fact a derivation of degree -1 of the exterior algebra of E_x^* , *i.e.*, for all p and $q \in \mathbb{Z}$, $\zeta \in \bigwedge^p E_x^*$, $\eta \in \bigwedge^q E_x^*$,

$$i(v)(\zeta \wedge \eta) = (i(v)\zeta) \wedge \eta + (-1)^p \zeta \wedge (i(v)\eta).$$

(iii) *The pairing between $\bigwedge E_x$ and $\bigwedge E_x^*$.* Let $x \in M$, p and $q \in \mathbb{Z}$, $\eta \in \bigwedge^p E_x^*$ and $v \in \bigwedge^q E_x$. We set

$$\langle \eta, v \rangle = \begin{cases} 0 & \text{if } p \neq q, \text{ or if } p < 0, \text{ or if } q < 0, \\ \eta v & \text{if } p = q = 0. \end{cases}$$

In order to define $\langle \eta, v \rangle$ when $p = q \geq 1$, let us first assume that η and v are decomposable, *i.e.*, that they can be written as

$$\eta = \eta_1 \wedge \dots \wedge \eta_p, \quad v = v_1 \wedge \dots \wedge v_p,$$

where $\eta_i \in E_x^*$, $v_j \in E_x$, $1 \leq i, j \leq p$. Then we set

$$\langle \eta, v \rangle = \det(\langle \eta_i, v_j \rangle),$$

One may see that $\langle \eta, v \rangle$ depends only on η and v , not on the way in which they are expressed as exterior products of elements of degree 1. The map $(\eta, v) \mapsto \langle \eta, v \rangle$ extends, in a unique way as a bilinear map

$$\bigwedge E_x^* \times \bigwedge E_x \rightarrow \mathbb{K}, \quad \text{still denoted by } (\eta, v) \mapsto \langle \eta, v \rangle,$$

called the *pairing*. That map allows us to consider each one of the two graded vector spaces $\bigwedge E_x^*$ and $\bigwedge E_x$ as the dual of the other one.

Let $\eta \in \bigwedge^p E_x^*$ and v_1, \dots, v_p be elements of E_x . The pairing $\langle \eta, v_1 \wedge \dots \wedge v_p \rangle$ is related, in a very simple way, to the value of η , considered as a p -multilinear form on E_x , on the set (v_1, \dots, v_p) . We have

$$\langle \eta, v_1 \wedge \dots \wedge v_p \rangle = \eta(v_1, \dots, v_p).$$

(iv) *The interior product of a form by a multivector.* For each $x \in M$ and $v \in E_x$, we have defined in 3.2.3 (ii) the interior product $i(v)$ as a derivation of degree -1 of the exterior algebra $\bigwedge E_x^*$ of forms at x . Let us now define, for each multivector $P \in \bigwedge E_x$, the interior product $i(P)$. Let us first assume that P is homogeneous of degree p , *i.e.*, that $P \in \bigwedge^p E_x$.

- For $p < 0$, $\bigwedge^p E_x = \{0\}$, therefore $i(P) = 0$.
- For $p = 0$, $\bigwedge^0 E_x = \mathbb{K}$, therefore P is a scalar and we set, for all $\eta \in \bigwedge E_x^*$,

$$i(P)\eta = P\eta.$$

- For $p \geq 1$ and $P \in \bigwedge^p E_x$ decomposable, *i.e.*,

$$P = P_1 \wedge \dots \wedge P_p, \quad \text{with } P_i \in E_x, \quad 1 \leq i \leq p,$$

we set

$$i(P_1 \wedge \dots \wedge P_p) = i(P_1) \circ \dots \circ i(P_p).$$

We see easily that $i(P)$ depends only of P , not of the way in which it is expressed as an exterior product of elements of degree 1.

- We extend by linearity the definition of $i(P)$ for all $P \in \bigwedge^p E_x$, and we see that $i(P)$ is a graded endomorphism of degree $-p$ of the graded vector space $\bigwedge E_x^*$. Observe that for $p \neq 1$, $i(P)$ is not in general a derivation of the exterior algebra $\bigwedge E_x^*$.

Finally, we extend by linearity the definition of $i(P)$ to all elements $P \in \bigwedge E_x$.

- (v) *The interior product by an exterior product.* It is easy to see that for all P and $Q \in \bigwedge E_x$,

$$i(P \wedge Q) = i(P) \circ i(Q).$$

- (vi) *Interior product and pairing.* For $p \in \mathbb{Z}$, $\eta \in \bigwedge^p E_x^*$ and $P \in \bigwedge^p E_x$, we have

$$i(P)\eta = (-1)^{(p-1)p/2} \langle \eta, P \rangle.$$

More generally, for p and $q \in \mathbb{Z}$, $P \in \bigwedge^p(E_x)$, $Q \in \bigwedge^q(E_x)$ and $\eta \in \bigwedge^{p+q}(E_x^*)$,

$$\langle i(P)\eta, Q \rangle = (-1)^{(p-1)p/2} \langle \eta, P \wedge Q \rangle.$$

This formula shows that the interior product by $P \in \bigwedge^p E_x$ is $(-1)^{(p-1)p/2}$ times the transpose, with respect to the pairing, of the exterior product by P on the left.

3.2.4 The exterior algebra of sections

Let (E, π, M) be a vector bundle of rank k on the field $\mathbb{K} = \mathbb{R}$ or \mathbb{C} , over a smooth manifold M , (E^*, ϖ, M) be its dual bundle and, for each integer $p \geq 1$, let $(\bigwedge^p E, \pi, M)$ and $(\bigwedge^p E^*, \varpi, M)$ be their respective p -th exterior powers.

For each $p \in \mathbb{Z}$, we will denote by $A^p(M, E)$ the space of smooth sections of $(\bigwedge^p E, \pi, M)$, *i.e.*, the space of smooth maps $Z : M \rightarrow \bigwedge^p E$ which satisfy

$$\pi \circ Z = \text{id}_M.$$

Similarly, for each $p \in \mathbb{Z}$, we will denote by $\Omega^p(M, E)$ the space of smooth sections of $(\bigwedge^p E^*, \varpi, M)$, *i.e.*, the space of smooth maps $\eta : M \rightarrow \bigwedge^p E^*$

which satisfy

$$\varpi \circ \eta = \text{id}_M .$$

Let us observe that $\Omega^p(M, E) = A^p(M, E^*)$.

We will denote by $A(M, E)$ and $\Omega(M, E)$ the direct sums

$$A(M, E) = \bigoplus_{p \in \mathbb{Z}} A^p(M, E), \quad \Omega(M, E) = \bigoplus_{p \in \mathbb{Z}} \Omega^p(M, E).$$

These direct sums, taken for all $p \in \mathbb{Z}$, are in fact taken for all integers p which satisfy $0 \leq p \leq k$, where k is the rank of the vector bundle (E, π, M) , since we have $A^p(M, E) = \Omega^p(M, E) = \{0\}$ for $p < 0$ as well as for $p > k$.

For $p = 0$, $A^0(M, E)$ and $\Omega^0(M, E)$ both coincide with the space $C^\infty(M, \mathbb{K})$ of smooth functions defined on M which take their values in the field \mathbb{K} .

Operations such as the exterior product, the interior product and the pairing, defined for each point $x \in M$ in 3.2.3, can be extended to elements in $A(M, E)$ and $\Omega(M, E)$.

(i) *The exterior product of two sections.* For example, the exterior product of two sections P and $Q \in A(M, E)$ is the section

$$x \in M, \quad x \mapsto (P \wedge Q)(x) = P(x) \wedge Q(x).$$

The exterior product of two sections η and $\zeta \in \Omega(M, E)$ is similarly defined.

With the exterior product as composition law, $A(M, E)$ and $\Omega(M, E)$ are \mathbb{Z} -graded associative and \mathbb{Z}_2 -commutative algebras, called the *algebra of multivectors* and the *algebra of forms* associated to the vector bundle (E, π, M) . Their subspaces $A^0(M, E)$ and $\Omega^0(M, E)$ of homogeneous elements of degree 0 both coincide with the usual algebra $C^\infty(M, \mathbb{K})$ of smooth \mathbb{K} -valued functions on M , with the usual product of functions as composition law. We observe that $A(M, E)$ and $\Omega(M, E)$ are \mathbb{Z} -graded modules over the ring of functions $C^\infty(M, \mathbb{K})$.

(ii) *The interior product by a section of $A(M, E)$.* For each $P \in A(M, E)$, the *interior product* $i(P)$ is an endomorphism of the graded vector space $\Omega(M, E)$.

If $p \in \mathbb{Z}$ and $P \in A^p(M, E)$, the endomorphism $i(P)$ is homogeneous of degree $-p$. For $p = 1$, $i(P)$ is a derivation of degree -1 of the algebra $\Omega(M, E)$.

(iii) *The pairing between $A(M, E)$ and $\Omega(M, E)$.* The pairing

$$(\eta, P) \mapsto \langle \eta, P \rangle, \quad \eta \in \Omega(M, E), \quad P \in A(M, E),$$

is a $C^\infty(M, \mathbb{K})$ -bilinear map, defined on $\Omega(M, E) \times A(M, E)$, which takes its values in $C^\infty(M, \mathbb{K})$.

4 Exterior powers of a Lie algebroid and of its dual

We consider now a Lie algebroid (E, π, M, ρ) over a smooth manifold M . We denote by (E^*, ϖ, M) its dual vector bundle, and use all the notations defined in Section 3. We will assume that the base field \mathbb{K} is \mathbb{R} , but most results remain valid for $\mathbb{K} = \mathbb{C}$. We will prove that differential operators such as the Lie derivative and the exterior derivative, which are well known for sections of the exterior powers of a tangent bundle and of its dual, still exist in this more general setting.

4.1 Lie derivatives with respect to sections of a Lie algebroid

We prove in this subsection that for each smooth section V of the Lie algebroid (E, π, M, ρ) , there exists a derivation of degree 0 of the exterior algebra $\Omega(M, E)$, called the *Lie derivative* with respect to V and denoted by $\mathcal{L}_\rho(V)$. When the Lie algebroid is the tangent bundle $(TM, \tau_M, M, \text{id}_{TM})$, we will recover the usual Lie derivative of differential forms with respect to a vector field.

4.1.1 Proposition. *Let (E, π, M, ρ) be a Lie algebroid on a smooth manifold M . For each smooth section $V \in A^1(M, E)$ of the vector bundle (E, π, M) , there exists a unique graded endomorphism of degree 0 of the graded algebra of*

exterior forms $\Omega(M, E)$, called the *Lie derivative with respect to V* and denoted by $\mathcal{L}_\rho(V)$, which satisfies the following properties:

(i) For a smooth function $f \in \Omega^0(M, E) = C^\infty(M, \mathbb{R})$,

$$\mathcal{L}_\rho(V)f = i(\rho \circ V)df = \mathcal{L}(\rho \circ V)f,$$

where $\mathcal{L}(\rho \circ V)$ denotes the usual Lie derivative with respect to the vector field $\rho \circ V$;

(ii) For a form $\eta \in \Omega^p(M, E)$ of degree $p > 0$, $\mathcal{L}_\rho(V)\eta$ is the form defined by the formula, where V_1, \dots, V_p are smooth sections of (E, π, M) ,

$$\begin{aligned} (\mathcal{L}_\rho(V)\eta)(V_1, \dots, V_p) &= \mathcal{L}_\rho(V)(\eta(V_1, \dots, V_p)) \\ &\quad - \sum_{i=1}^p \eta(V_1, \dots, V_{i-1}, \{V, V_i\}, V_{i+1}, \dots, V_p). \end{aligned}$$

Proof:

Clearly (i) defines a function $\mathcal{L}_\rho(V)f \in \Omega^0(M, E) = C^\infty(M, \mathbb{R})$. We see immediately that for f and $g \in C^\infty(M, \mathbb{R})$,

$$\mathcal{L}_\rho(V)(fg) = (\mathcal{L}_\rho(V)f)g + f(\mathcal{L}_\rho(V)g). \quad (*)$$

Now (ii) defines a map $(V_1, \dots, V_p) \mapsto (\mathcal{L}_\rho(V)\eta)(V_1, \dots, V_p)$ on $(A^1(M, E))^p$, with values in $C^\infty(M, \mathbb{R})$. In order to prove that this map defines an element $\mathcal{L}_\rho(V)\eta$ in $\Omega^p(M, E)$, it is enough to prove that it is skew-symmetric and $C^\infty(M, \mathbb{R})$ -linear in each argument. The skew-symmetry and the \mathbb{R} -linearity in each argument are easily verified. There remains only to prove that for each function $f \in C^\infty(M, \mathbb{R})$,

$$(\mathcal{L}_\rho(V)\eta)(fV_1, V_2, \dots, V_p) = f(\mathcal{L}_\rho(V)\eta)(V_1, V_2, \dots, V_p). \quad (**)$$

We have

$$\begin{aligned} (\mathcal{L}_\rho(V)\eta)(fV_1, V_2, \dots, V_p) &= \mathcal{L}_\rho(V)(\eta(fV_1, V_2, \dots, V_p)) \\ &\quad - \eta(\{V, fV_1\}, V_2, \dots, V_p) \\ &\quad - \sum_{i=2}^p \eta(fV_1, V_2, \dots, V_{i-1}, \{V, V_i\}, V_{i+1}, \dots, V_p). \end{aligned}$$

By using (*), we may write

$$\begin{aligned} \mathcal{L}_\rho(V)(\eta(fV_1, V_2, \dots, V_p)) &= \mathcal{L}_\rho(V)(f\eta(V_1, V_2, \dots, V_p)) \\ &= (\mathcal{L}_\rho(V)f)\eta(V_1, V_2, \dots, V_p) \\ &\quad + f\mathcal{L}_\rho(V)(\eta(V_1, V_2, \dots, V_p)). \end{aligned}$$

Using the property of the anchor, we also have

$$\{V, fV_1\} = (i(\rho \circ V)df)V_1 + f\{V, V_1\} = (\mathcal{L}_\rho(V)f)V_1 + f\{V, V_1\}.$$

Equality (**) follows immediately.

The endomorphism $\mathcal{L}_\rho(V)$, defined on the subspaces of homogeneous forms, can then be extended, in a unique way, to $\Omega(M, E)$, by imposing the \mathbb{R} -linearity of the map $\eta \mapsto \mathcal{L}_\rho(V)\eta$. \bowtie

Let us now introduce the $\Omega(M, E)$ -valued exterior derivative of a function. In the next section, that definition will be extended to all elements in $\Omega(M, E)$.

4.1.2 Definition. Let (E, π, M, ρ) be a Lie algebroid on a smooth manifold M . For each function $f \in \Omega^0(M, E) = C^\infty(M, \mathbb{R})$, we call $\Omega(M, E)$ -valued exterior derivative of f , and denote by $d_\rho f$, the unique element in $\Omega^1(M, E)$ such that, for each section $V \in A^1(M, E)$,

$$\langle d_\rho f, V \rangle = \langle df, \rho \circ V \rangle.$$

4.1.3 Remark. Let us observe that the transpose of the anchor $\rho : E \rightarrow TM$ is a vector bundle map ${}^t\rho : T^*M \rightarrow E^*$. By composition of that map with the usual differential of functions, we obtain the $\Omega(M, E)$ -valued exterior differential d_ρ . We have indeed

$$d_\rho f = {}^t\rho \circ df.$$

4.1.4 Proposition. Under the assumptions of Proposition 4.1.1, the Lie derivative has the following properties.

1. For each $V \in A^1(M, E)$ and $f \in C^\infty(M, \mathbb{R})$,

$$\mathcal{L}_\rho(V)(d_\rho f) = d_\rho(\mathcal{L}_\rho(V)f).$$

2. For each V and $W \in A^1(M, E)$, $\eta \in \Omega(M, E)$,

$$i(\{V, W\})\eta = (\mathcal{L}_\rho(V) \circ i(W) - i(W) \circ \mathcal{L}_\rho(V))\eta.$$

3. For each $V \in A^1(M, E)$, $\mathcal{L}_\rho(V)$ is a derivation of degree 0 of the exterior algebra $\Omega(M, E)$. That means that for all η and $\zeta \in \Omega(M, E)$,

$$\mathcal{L}_\rho(V)(\eta \wedge \zeta) = (\mathcal{L}_\rho(V)\eta) \wedge \zeta + \eta \wedge (\mathcal{L}_\rho(V)\zeta).$$

4. For each V and $W \in A^1(M, E)$, $\eta \in \Omega(M, E)$,

$$\mathcal{L}_\rho(\{V, W\})\eta = (\mathcal{L}_\rho(V) \circ \mathcal{L}_\rho(W) - \mathcal{L}_\rho(W) \circ \mathcal{L}_\rho(V))\eta.$$

5. For each $V \in A^1(M, E)$, $f \in C^\infty(M, \mathbb{R})$ and $\eta \in \Omega(M, E)$,

$$\mathcal{L}_\rho(fV)\eta = f\mathcal{L}_\rho(V)\eta + d_\rho f \wedge i(V)\eta.$$

Proof:

1. Let $W \in A^1(M, E)$. Then

$$\begin{aligned} \langle \mathcal{L}_\rho(V)(d_\rho f), W \rangle &= \mathcal{L}_\rho(V)\langle d_\rho f, W \rangle - \langle d_\rho f, \{V, W\} \rangle \\ &= \mathcal{L}(\rho \circ V) \circ \mathcal{L}(\rho \circ W)f - \mathcal{L}(\rho \circ \{V, W\})f \\ &= \mathcal{L}(\rho \circ W) \circ \mathcal{L}(\rho \circ V)f \\ &= \langle d_\rho(\mathcal{L}_\rho(V)f), W \rangle, \end{aligned}$$

so Property 1 is proven.

2. Let V and $W \in A^1(M, E)$, $\eta \in \Omega^p(M, E)$, $V_1, \dots, V_{p-1} \in A^1(M, E)$. We may write

$$\begin{aligned} &(\mathcal{L}_\rho(V) \circ i(W)\eta)(V_1, \dots, V_{p-1}) \\ &= \mathcal{L}_\rho(V)(\eta(W, V_1, \dots, V_{p-1})) \\ &\quad - \sum_{k=1}^{p-1} \eta(W, V_1, \dots, V_{k-1}, \{V, V_k\}, V_{k+1}, \dots, V_{p-1}) \\ &= (\mathcal{L}_\rho(V)\eta)(W, V_1, \dots, V_{p-1}) + \eta(\{V, W\}, V_1, \dots, V_{p-1}) \\ &= \left((i(W) \circ \mathcal{L}_\rho(V) + i(\{V, W\}))\eta \right)(V_1, \dots, V_{p-1}), \end{aligned}$$

so Property 2 is proven.

3. Let $V \in A^1(M, E)$, $\eta \in \Omega^p(M, E)$ and $\zeta \in \Omega^q(M, E)$. For $p < 0$, as well as for $q < 0$, both sides of the equality stated in Property 3 vanish, so that equality is trivially satisfied. For $p = q = 0$, that equality is also satisfied, as shown by Equality (*) in the proof of Proposition 4.1.1. We still have to prove that equality for $p > 0$ and (or) $q > 0$. We will do that by induction on the value of $p + q$. Let $r \geq 1$ be an integer such that the equality stated in Property 3 holds for $p + q \leq r - 1$. Such an integer exists, for example $r = 1$. We assume now that $p \geq 0$ and $q \geq 0$ are such that $p + q = r$. Let $W \in A^1(M, E)$. By using Property 2, we may write

$$\begin{aligned} i(W) \circ \mathcal{L}_\rho(V)(\eta \wedge \zeta) &= \mathcal{L}_\rho(V) \circ i(W)(\eta \wedge \zeta) - i(\{V, W\})(\eta \wedge \zeta) \\ &= \mathcal{L}_\rho(V)(i(W)\eta \wedge \zeta + (-1)^p \eta \wedge i(W)\zeta) \\ &\quad - i(\{V, W\})\eta \wedge \zeta - (-1)^p \eta \wedge i(\{V, W\})\zeta. \end{aligned}$$

Since $i(W)\eta \in \Omega^{p-1}(M, E)$ and $i(W)\zeta \in \Omega^{q-1}(M, E)$, the induction assumption allows us to use Property 3 to transform the first terms of the right hand side. We obtain

$$\begin{aligned} i(W) \circ \mathcal{L}_\rho(V)(\eta \wedge \zeta) &= (\mathcal{L}_\rho(V) \circ i(W)\eta) \wedge \zeta + i(W)\eta \wedge \mathcal{L}_\rho(V)\zeta \\ &\quad + (-1)^p (\mathcal{L}_\rho(V)\eta) \wedge i(W)\zeta + (-1)^p \eta \wedge (\mathcal{L}_\rho(V) \circ i(W)\zeta) \\ &\quad - i(\{V, W\})\eta \wedge \zeta - (-1)^p \eta \wedge i(\{V, W\})\zeta. \end{aligned}$$

By rearranging the terms, we obtain

$$\begin{aligned} i(W) \circ \mathcal{L}_\rho(V)(\eta \wedge \zeta) &= (\mathcal{L}_\rho(V) \circ i(W)\eta - i(\{V, W\})\eta) \wedge \zeta \\ &\quad + (-1)^p \eta \wedge (\mathcal{L}_\rho(V) \circ i(W)\zeta - i(\{V, W\})\zeta) \\ &\quad + i(W)\eta \wedge \mathcal{L}_\rho(V)\zeta + (-1)^p (\mathcal{L}_\rho(V)\eta) \wedge i(W)\zeta. \end{aligned}$$

By using again Property 2 we get

$$\begin{aligned} i(W) \circ \mathcal{L}_\rho(V)(\eta \wedge \zeta) &= (i(W) \circ \mathcal{L}_\rho(V)\eta) \wedge \zeta + (-1)^p \eta \wedge (i(W) \circ \mathcal{L}_\rho(V)\zeta) \\ &\quad + i(W)\eta \wedge \mathcal{L}_\rho(V)\zeta + (-1)^p \mathcal{L}_\rho(V)\eta \wedge i(W)\zeta \\ &= i(W)(\mathcal{L}_\rho(V)\eta \wedge \zeta + \eta \wedge \mathcal{L}_\rho(V)\zeta). \end{aligned}$$

Since that last equality holds for all $W \in A^1(M, E)$, it follows that Property 3 holds for $\eta \in \Omega^p(M, E)$ and $\zeta \in \Omega^q(M, E)$, with $p \geq 0$, $q \geq 0$ and $p + q = r$. We have thus proven by induction that Property 3 holds for all p and $q \in \mathbb{Z}$, $\eta \in \Omega^p(M, E)$, $\zeta \in \Omega^q(M, E)$. The same equality holds, by bilinearity, for all η and $\zeta \in \Omega(M, E)$.

4. Let V and $W \in A^1(M, E)$. Then $\{V, W\} \in A^1(M, E)$ and, by Property 3, $\mathcal{L}_\rho(V)$, $\mathcal{L}_\rho(W)$ and $\mathcal{L}_\rho(\{V, W\})$ are derivations of degree 0 of the graded algebra $\Omega(M, E)$. By 3.1.6 (ii), the graded bracket

$$[\mathcal{L}_\rho(V), \mathcal{L}_\rho(W)] = \mathcal{L}_\rho(V) \circ \mathcal{L}_\rho(W) - \mathcal{L}_\rho(W) \circ \mathcal{L}_\rho(V)$$

is also a derivation of degree 0 of $\Omega(M, E)$. Property 4 means that the derivations $\mathcal{L}_\rho(\{V, W\})$ and $[\mathcal{L}_\rho(V), \mathcal{L}_\rho(W)]$ are equal. In order to prove that equality, it is enough to prove that it holds true for $\eta \in \Omega^0(M, E)$ and for $\eta \in \Omega^1(M, E)$, since the graded algebra $\Omega(M, E)$ is generated by its homogeneous elements of degrees 0 and 1.

Let $f \in \Omega^0(M, E) = C^\infty(M, \mathbb{R})$. We have

$$\begin{aligned} \mathcal{L}_\rho(\{V, W\})f &= \mathcal{L}(\rho \circ \{V, W\})f \\ &= \mathcal{L}([\rho \circ V, \rho \circ W])f \\ &= [\mathcal{L}(\rho \circ V), \mathcal{L}(\rho \circ W)]f \\ &= [\mathcal{L}_\rho(V), \mathcal{L}_\rho(W)]f, \end{aligned}$$

therefore Property 4 holds for $\eta = f \in \Omega^0(M, E)$.

Now let $\eta \in \Omega^1(M, E)$ and $Z \in A^1(M, E)$. By using Property 2, then Property 4 for elements $\eta \in \Omega^0(M, E)$, we may write

$$\begin{aligned} i(Z) \circ \mathcal{L}_\rho(\{V, W\})\eta &= \mathcal{L}_\rho(\{V, W\})(i(Z)\eta) - i(\{\{V, W\}, Z\})\eta \\ &= (\mathcal{L}_\rho(V) \circ \mathcal{L}_\rho(W) - \mathcal{L}_\rho(W) \circ \mathcal{L}_\rho(V))(i(Z)\eta) \\ &\quad - i(\{\{V, W\}, Z\})\eta. \end{aligned}$$

By using Property 2 and the Jacobi identity, we obtain

$$\begin{aligned}
i(Z) \circ \mathcal{L}_\rho(\{V, W\})\eta &= \mathcal{L}_\rho(V)(i(\{W, Z\})\eta + i(Z) \circ \mathcal{L}_\rho(W)\eta) \\
&\quad - \mathcal{L}_\rho(W)(i(\{V, Z\})\eta + i(Z) \circ \mathcal{L}_\rho(V)\eta) \\
&\quad - i(\{\{V, W\}, Z\})\eta \\
&= i(\{W, Z\})\mathcal{L}_\rho(V)\eta + i(\{V, Z\})\mathcal{L}_\rho(W)\eta \\
&\quad - i(\{V, Z\})\mathcal{L}_\rho(W)\eta - i(\{W, Z\})\mathcal{L}_\rho(V)\eta \\
&\quad + i(Z) \circ (\mathcal{L}_\rho(V) \circ \mathcal{L}_\rho(W) - \mathcal{L}_\rho(W) \circ \mathcal{L}_\rho(V))\eta \\
&\quad + i(\{V, \{W, Z\}\} - \{W, \{V, Z\}\} - \{\{V, W\}, Z\})\eta \\
&= i(Z) \circ (\mathcal{L}_\rho(V) \circ \mathcal{L}_\rho(W) - \mathcal{L}_\rho(W) \circ \mathcal{L}_\rho(V))\eta.
\end{aligned}$$

Since that last equality holds for all $Z \in A^1(M, E)$, Property 4 holds for all $\eta \in \Omega^1(M, E)$, and therefore for all $\eta \in \Omega(M, E)$.

5. Let $V \in A^1(M, E)$ and $f \in C^\infty(M, \mathbb{R})$. We have seen (Property 4) that $\mathcal{L}_\rho(fV)$ is a derivation of degree 0 of $\Omega(M, E)$. We easily verify that

$$\eta \mapsto f\mathcal{L}_\rho(V)\eta + d_\rho f \wedge i(V)\eta$$

is too a derivation of degree 0 of $\Omega(M, E)$. Property 5 means that these two derivations are equal. As above, it is enough to prove that Property 5 holds for $\eta \in \Omega^0(M, E)$ and for $\eta \in \Omega^1(M, E)$.

Let $g \in \Omega^0(M, E) = C^\infty(M, \mathbb{R})$. We may write

$$\mathcal{L}_\rho(fV)g = i(fV)d_\rho g = f\mathcal{L}_\rho(V)g,$$

which shows that Property 5 holds for $\eta = g \in \Omega^0(M, E)$.

Let $\eta \in \Omega^1(M, E)$, and $W \in A^1(M, E)$. We have

$$\begin{aligned}
\langle \mathcal{L}_\rho(fV)\eta, W \rangle &= \mathcal{L}_\rho(fV)(\langle \eta, W \rangle) - \langle \eta, \{fV, W\} \rangle \\
&= f\mathcal{L}_\rho(V)(\langle \eta, W \rangle) - f\langle \eta, \{V, W\} \rangle + \langle \eta, (i(W)d_\rho f)V \rangle \\
&= \langle f\mathcal{L}_\rho(V)\eta, W \rangle + (i(W)d_\rho f)i(V)\eta \\
&= \langle f\mathcal{L}_\rho(V)\eta + d_\rho f \wedge i(V)\eta, W \rangle,
\end{aligned}$$

since, η being in $\Omega^1(M, E)$, $i(W) \circ i(V)\eta = 0$. The last equality being satisfied for all $W \in A^1(M, E)$, the result follows. \bowtie

The next Proposition shows that for each $V \in A^1(M, E)$, the Lie derivative $\mathcal{L}_\rho(V)$, already defined as a derivation of degree 0 of the graded algebra $\Omega(M, E)$, can also be extended into a derivation of degree 0 of the graded algebra $A(M, E)$, with very nice properties. As we will soon see, the Schouten-Nijenhuis bracket will appear as a very natural further extension of the Lie derivative.

4.1.5 Proposition. *Let (E, π, M, ρ) be a Lie algebroid on a smooth manifold M . For each smooth section $V \in A^1(M, E)$ of the vector bundle (E, π, M) , there exists a unique graded endomorphism of degree 0 of the graded algebra of multivectors $A(M, E)$, called the Lie derivative with respect to V and denoted by $\mathcal{L}_\rho(V)$, which satisfies the following properties:*

- (i) For a smooth function $f \in A^0(M, E) = C^\infty(M, \mathbb{R})$,

$$\mathcal{L}_\rho(V)f = i(\rho \circ V)df = \mathcal{L}(\rho \circ V)f,$$

where $\mathcal{L}(\rho \circ V)$ denotes the usual Lie derivative with respect to the vector field $\rho \circ V$;

- (ii) For an integer $p \geq 1$ and a multivector $P \in A^p(M, E)$, $\mathcal{L}_\rho(V)P$ is the unique element in $A^p(M, E)$ such that, for all $\eta \in \Omega^p(M, E)$,

$$\langle \eta, \mathcal{L}_\rho(V)P \rangle = \mathcal{L}_\rho(V)(\langle \eta, P \rangle) - \langle \mathcal{L}_\rho(V)\eta, P \rangle.$$

Proof:

Let us first observe that $A^0(M, E) = \Omega^0(M, E) = C^\infty(M, \mathbb{R})$, and that for $f \in A^0(M, E)$, the definition of $\mathcal{L}_\rho(V)f$ given above is the same as that given in Proposition 4.1.1.

Now let $p \geq 1$ and $P \in A^p(M, E)$. The map

$$\eta \mapsto K(\eta) = \mathcal{L}_\rho(V)(\langle \eta, P \rangle) - \langle \mathcal{L}_\rho(V)\eta, P \rangle$$

is clearly an \mathbb{R} -linear map defined on $\Omega^p(M, E)$, with values in $C^\infty(M, \mathbb{R})$. Let $f \in C^\infty(M, \mathbb{R})$. We have

$$\begin{aligned} K(f\eta) &= \mathcal{L}_\rho(V)(\langle f\eta, P \rangle) - \langle \mathcal{L}_\rho(V)(f\eta), P \rangle \\ &= f \left(\mathcal{L}_\rho(V)(\langle \eta, P \rangle) - \langle \mathcal{L}_\rho(V)\eta, P \rangle \right) \\ &\quad + (\mathcal{L}_\rho(V)f)\langle \eta, P \rangle - (\mathcal{L}_\rho(V)f)\langle \eta, P \rangle \\ &= fK(\eta). \end{aligned}$$

This proves that the map K is $C^\infty(M, \mathbb{R})$ -linear. Since the pairing allows us to consider the vector bundle $(\wedge^p E, \pi, M)$ as the dual of $(\wedge^p E^*, \varpi, M)$, we see that there exists a unique element $\mathcal{L}_\rho(V)P \in A^p(M, E)$ such that, for all $\eta \in \Omega^p(M, E)$,

$$K(\eta) = \mathcal{L}_\rho(V)(\langle \eta, P \rangle) - \langle \mathcal{L}_\rho(V)\eta, P \rangle = \langle \eta, \mathcal{L}_\rho(V)P \rangle,$$

and that ends the proof. \boxtimes

4.1.6 Proposition. *Under the assumptions of Proposition 4.1.5, the Lie derivative has the following properties.*

1. For each V and $W \in A^1(M, E)$,

$$\mathcal{L}_\rho(V)(W) = \{V, W\}.$$

2. For $V, V_1, \dots, V_p \in A^1(M, E)$,

$$\mathcal{L}_\rho(V)(V_1 \wedge \dots \wedge V_p) = \sum_{i=1}^p V_1 \wedge \dots \wedge V_{i-1} \wedge \{V, V_i\} \wedge V_{i+1} \wedge \dots \wedge V_p.$$

3. For each $V \in A^1(M, E)$, $\mathcal{L}_\rho(V)$ is a derivation of degree 0 of the exterior algebra $A(M, E)$. That means that for all P and $Q \in A(M, E)$,

$$\mathcal{L}_\rho(V)(P \wedge Q) = (\mathcal{L}_\rho(V)P) \wedge Q + P \wedge \mathcal{L}_\rho(V)Q.$$

4. For each $V \in A^1(M, E)$, $P \in A(M, E)$ and $\eta \in \Omega(M, E)$,

$$\mathcal{L}_\rho(V)(i(P)\eta) = i(\mathcal{L}_\rho(V)P)\eta + i(P)(\mathcal{L}_\rho(V)\eta).$$

5. Similarly, for each $V \in A^1(M, E)$, $P \in A(M, E)$ and $\eta \in \Omega(M, E)$,

$$\mathcal{L}_\rho(V)(\langle \eta, P \rangle) = \langle \mathcal{L}_\rho(V)\eta, P \rangle + \langle \eta, \mathcal{L}_\rho(V)P \rangle.$$

6. For each V and $W \in A^1(M, E)$, $P \in A(M, E)$,

$$\mathcal{L}_\rho(\{V, W\})P = (\mathcal{L}_\rho(V) \circ \mathcal{L}_\rho(W) - \mathcal{L}_\rho(W) \circ \mathcal{L}_\rho(V))P.$$

7. For each $V \in A^1(M, E)$, $f \in C^\infty(M, \mathbb{R})$, $P \in A(M, E)$ and $\eta \in \Omega(M, E)$,

$$\langle \eta, \mathcal{L}_\rho(fV)P \rangle = f\langle \mathcal{L}_\rho(V)P, \eta \rangle + \langle d_\rho f \wedge i(V)\eta, P \rangle.$$

Proof:

1. Let V and $W \in A^1(M, E)$, $\eta \in \Omega(M, E)$. We may write

$$\begin{aligned} \langle \eta, \mathcal{L}_\rho(V)W \rangle &= \mathcal{L}_\rho(V)(\langle \eta, W \rangle) - \langle \mathcal{L}_\rho(V)\eta, W \rangle \\ &= \langle \eta, \{V, W\} \rangle. \end{aligned}$$

We have proven Property 1.

2. The proof is similar to that of Property 1.

3. When $P = V_1 \wedge \cdots \wedge V_p$ and $Q = W_1 \wedge \cdots \wedge W_q$ are decomposable homogeneous elements in $A(M, E)$, Property 3 is an easy consequence of 2. The validity of Property 3 for all P and $Q \in A(M, E)$ follows by linearity.

4. When $P = V_1 \wedge \cdots \wedge V_p$ is a decomposable homogeneous element in $A(M, E)$, Property 4 is an easy consequence of Property 2. The validity of Property 4 for all P and $Q \in A(M, E)$ follows by linearity.

5. This is an immediate consequence of Property 4.

6. This is an immediate consequence of Property 4 of this Proposition and of Property 4 of Proposition 4.1.4.

7. This is an immediate consequence of Property 4 of this Proposition and of Property 5 of Proposition 4.1.4. \bowtie

4.2 The $\Omega(M, E)$ -valued exterior derivative

We have introduced above (Definition 4.1.2) the $\Omega(M, E)$ -valued exterior derivative of a function $f \in \Omega^0(M, E) = C^\infty(M, \mathbb{R})$. The next proposition shows that the $\Omega(M, E)$ -valued exterior derivative extends as a graded endomorphism of degree 1 of the graded algebra $\Omega(M, E)$. We will see later (Proposition 4.2.3) that the $\Omega(M, E)$ -valued exterior derivative is in fact a derivation of degree 1 of $\Omega(M, E)$.

4.2.1 Proposition. *Let (E, π, M, ρ) be a Lie algebroid over a smooth manifold M . There exists a unique graded endomorphism of degree 1 of the exterior algebra of forms $\Omega(M, E)$, called the $\Omega(M, E)$ -valued exterior derivative (or, in brief, the exterior derivative) and denoted by d_ρ , which satisfies the following properties:*

(i) For $f \in \Omega^0(M, E) = C^\infty(M, \mathbb{R})$, $d_\rho f$ is the unique element in $\Omega^1(M, E)$, already defined (Definition 4.1.2), such that, for each $V \in A^1(M, E)$,

$$\langle d_\rho f, V \rangle = \mathcal{L}_\rho(V)f = \langle df, \rho \circ V \rangle = \langle {}^t\rho \circ df, V \rangle,$$

where d stands for the usual exterior derivative of smooth functions on M , and ${}^t\rho : E^* \rightarrow T^*M$ is the transpose of the anchor ρ .

(ii) For $p \geq 1$ and $\eta \in \Omega^p(M, E)$, $d_\rho \eta$ is the unique element in $\Omega^{p+1}(M, E)$ such that, for all $V_0, \dots, V_p \in A^1(M, E)$,

$$\begin{aligned} d_\rho \eta(V_0, \dots, V_p) &= \sum_{i=0}^p (-1)^i \mathcal{L}_\rho(V_i) (\eta(V_0, \dots, \widehat{V}_i, \dots, V_p)) \\ &\quad + \sum_{0 \leq i < j \leq p} (-1)^{i+j} \eta(\{V_i, V_j\}, V_0, \dots, \widehat{V}_i, \dots, \widehat{V}_j, \dots, V_p), \end{aligned}$$

where the symbol $\widehat{}$ over the terms V_i and V_j means that these terms are omitted.

Proof:

For $f \in \Omega^0(M, E)$, $d_\rho f$ is clearly an element in $\Omega^1(M, E)$.

Let $p \geq 1$ and $\eta \in \Omega^p(M, E)$. As defined in (ii), $d_\rho \eta$ is a map, defined on $(A^1(M, E))^p$, with values in $C^\infty(M, \mathbb{R})$. The reader will immediately see that this map is skew-symmetric and \mathbb{R} -linear in each of its arguments. In order to prove that $d_\rho \eta$ is an element in $\Omega^{p+1}(M, E)$, it remains only to verify that as a map, $d_\rho \eta$ is $C^\infty(M, \mathbb{R})$ -linear in each of its arguments, or simply in its first argument, since the skew-symmetry will imply the same property for all other arguments. Let $f \in C^\infty(M, \mathbb{R})$. We have

$$\begin{aligned} d_\rho \eta(fV_0, V_1, \dots, V_p) &= \mathcal{L}_\rho(fV_0)(\eta(V_1, \dots, V_p)) \\ &+ \sum_{i=1}^p (-1)^i \mathcal{L}_\rho(V_i)(f\eta(V_0, \dots, \widehat{V}_i, \dots, V_p)) \\ &+ \sum_{1 \leq j \leq p} (-1)^j \eta(\{fV_0, V_j\}, V_1, \dots, \widehat{V}_j, \dots, V_p) \\ &+ \sum_{1 \leq i < j \leq p} (-1)^{i+j} \eta(\{V_i, V_j\}, fV_0, V_1, \dots, \widehat{V}_i, \dots, \widehat{V}_j, \dots, V_p). \end{aligned}$$

By a rearrangement of the terms in the right hand side, and by using the formulae

$$\mathcal{L}_\rho(V_i)(f\eta(\dots)) = (\mathcal{L}_\rho(V_i)f)\eta(\dots) + f\mathcal{L}_\rho(V_i)(\eta(\dots)),$$

and

$$\{fV_0, V_j\} = f\{V_0, V_j\} - (\mathcal{L}_\rho(V_j)f)V_0,$$

we obtain

$$d_\rho \eta(fV_0, V_1, \dots, V_p) = fd_\rho \eta(V_0, V_1, \dots, V_p).$$

We have shown that $d_\rho \eta \in \Omega^{p+1}(M, E)$. The $\Omega(M, E)$ -valued exterior derivative so defined on $\Omega^p(M, E)$ for all $p \in \mathbb{Z}$ extends, in a unique way, into a graded endomorphism of degree 1 of $\Omega(M, E)$. \bowtie

4.2.2 Remark. Let $p \geq 1$, $\eta \in \Omega^p(M, E)$ and $V_0, \dots, V_p \in A^1(M, E)$. The formula for $d_\rho \eta$ given in Proposition 4.2.1 can be cast into another form, often

useful:

$$d_\rho \eta(V_0, \dots, V_p) = \sum_{i=0}^p (-1)^i (\mathcal{L}_\rho(V_i) \eta)(V_0, \dots, \widehat{V}_i, \dots, V_p) \\ - \sum_{0 \leq i < j \leq p} (-1)^{i+j} \eta(\{V_i, V_j\}, V_0, \dots, \widehat{V}_i, \dots, \widehat{V}_j, \dots, V_p),$$

where the symbol $\widehat{}$ over the terms V_i and V_j means that these terms are omitted.

For example, for $p = 1$,

$$d_\rho \eta(V_0, V_1) = \mathcal{L}_\rho(V_0)(\eta(V_1)) - \mathcal{L}_\rho(V_1)(\eta(V_0)) - \eta(\{V_0, V_1\}) \\ = \langle \mathcal{L}_\rho(V_0) \eta, V_1 \rangle - \langle \mathcal{L}_\rho(V_1) \eta, V_0 \rangle + \eta(\{V_0, V_1\}).$$

4.2.3 Proposition. *Under the assumptions of Proposition 4.2.1, the $\Omega(M, E)$ -valued exterior derivative has the following properties:*

1. *Let $V \in A^1(M, E)$. The Lie derivative $\mathcal{L}_\rho(V)$, the exterior derivative d_ρ and the interior product $i(V)$ are related by the formula*

$$\mathcal{L}_\rho(V) = i(V) \circ d_\rho + d_\rho \circ i(V).$$

2. *The exterior derivative d_ρ is a derivation of degree 1 of the exterior algebra $\Omega(M, E)$. That means that for each $p \in \mathbb{Z}$, $\eta \in \Omega^p(M, E)$ and $\zeta \in \Omega(M, E)$,*

$$d_\rho(\eta \wedge \zeta) = d_\rho \eta \wedge \zeta + (-1)^p \eta \wedge d_\rho \zeta.$$

3. *Let $V \in A^1(M, E)$. Then*

$$\mathcal{L}_\rho(V) \circ d_\rho = d_\rho \circ \mathcal{L}_\rho(V).$$

4. *The square of d_ρ vanishes identically,*

$$d_\rho \circ d_\rho = 0.$$

Proof:

1. Let $V_0 = V, V_1, \dots, V_p \in A^1(M, E), \eta \in \Omega^p(M, E)$. Then

$$\begin{aligned} (i(V) \circ d_\rho \eta)(V_1, \dots, V_p) &= d_\rho \eta(V, V_1, \dots, V_p) \\ &= \sum_{i=0}^p (-1)^i \mathcal{L}_\rho(V_i)(\eta(V_0, \dots, \widehat{V}_i, \dots, V_p)) \\ &\quad + \sum_{0 \leq i < j \leq p} (-1)^{i+j} \eta(\{V_i, V_j\}, V_0, \dots, \widehat{V}_i, \dots, \widehat{V}_j, \dots, V_p), \end{aligned}$$

and

$$\begin{aligned} (d_\rho \circ i(V)\eta)(V_1, \dots, V_p) &= \sum_{i=1}^p (-1)^{i-1} \mathcal{L}_\rho(V_i)(\eta(V_0, \dots, \widehat{V}_i, \dots, V_p)) \\ &\quad + \sum_{1 \leq i < j \leq p} (-1)^{i+j} \eta(V_0, \{V_i, V_j\}, V_1, \dots, \widehat{V}_i, \dots, \widehat{V}_j, \dots, V_p). \end{aligned}$$

Let us add these two equalities. Several terms cancel, and we obtain

$$\begin{aligned} &\left((i(V) \circ d_\rho + d_\rho \circ i(V)) \eta \right)(V_1, \dots, V_p) \\ &= \mathcal{L}_\rho(V_0)(\eta(V_1, \dots, V_p)) + \sum_{j=1}^p (-1)^j \eta(\{V_0, V_j\}, V_1, \dots, \widehat{V}_j, \dots, V_p). \end{aligned}$$

When we shift, in the last term of the right hand side, the argument $\{V_0, V_j\}$ to the j -th position, we obtain

$$\begin{aligned} &\left((i(V) \circ d_\rho + d_\rho \circ i(V)) \eta \right)(V_1, \dots, V_p) \\ &= \mathcal{L}_\rho(V_0)(\eta(V_1, \dots, V_p)) + \sum_{j=1}^p \eta(V_1, \dots, V_{j-1}, \{V_0, V_j\}, V_{j+1}, \dots, V_p) \\ &= (\mathcal{L}_\rho(V_0)\eta)(V_1, \dots, V_p). \end{aligned}$$

2. For $\eta = f$ and $\zeta = g \in \Omega^0(M, E) = C^\infty(M, \mathbb{R})$, Property 2 holds since we have, for all $V \in A^1(M, E)$,

$$\begin{aligned} \langle d_\rho(fg), V \rangle &= \langle d(fg), \rho \circ V \rangle = \langle f dg + g df, \rho \circ V \rangle \\ &= \langle f d_\rho g + g d_\rho f, V \rangle. \end{aligned}$$

Now let $p \geq 0$ and $q \geq 0$ be two integers, $\eta \in \Omega^p(M, E)$, $\zeta \in \Omega^q(M, E)$. We will prove that Property 2 holds by induction on $p + q$. Just above, we have seen that it holds for $p + q = 0$. Let us assume that r is an integer such that Property 2 holds for $p + q \leq r$, and that now $p + q = r + 1$. Let $V \in A^1(M, E)$. We may write

$$\begin{aligned} i(V)d_\rho(\eta \wedge \zeta) &= \mathcal{L}_\rho(V)(\eta \wedge \zeta) - d_\rho \circ i(V)(\eta \wedge \zeta) \\ &= \mathcal{L}_\rho(V)\eta \wedge \zeta + \eta \wedge \mathcal{L}_\rho(V)\zeta \\ &\quad - d_\rho(i(V)\eta \wedge \zeta + (-1)^p \eta \wedge i(V)\zeta). \end{aligned}$$

We may now use the induction assumption, since in the last terms of the right hand side $i(V)\eta \in \Omega^{p-1}(M, E)$ and $i(V)\zeta \in \Omega^{q-1}(M, E)$. After some rearrangements of the terms we obtain

$$i(V)d_\rho(\eta \wedge \zeta) = i(V)(d_\rho \eta \wedge \zeta + \eta \wedge d_\rho \zeta).$$

Since that result holds for all $V \in A^1(M, E)$, Property 2 holds for $p + q = r + 1$, and therefore for all p and $q \in \mathbb{Z}$.

3. Let $V \in A^1(M, E)$. We know (Proposition 4.1.4) that $\mathcal{L}_\rho(V)$ is a derivation of degree 0 of the exterior algebra $\Omega(M, E)$, and we have just seen (Property 2) that d_ρ is a derivation of degree 1 of that algebra. Therefore, by 3.1.6, their graded bracket

$$[\mathcal{L}_\rho(V), d_\rho] = \mathcal{L}_\rho(V) \circ d_\rho - d_\rho \circ \mathcal{L}_\rho(V)$$

is a derivation of degree 1 of $\Omega(M, E)$. In order to prove that that derivation is equal to 0, it is enough to prove that it vanishes on $\Omega^0(M, E)$ and on $\Omega^1(M, E)$. We have already proven that it vanishes on $\Omega^0(M, E)$ (Property 1 of 4.1.4). Let $\eta \in \Omega^1(M, E)$ and $W \in A^1(M, E)$. By using Property 1 of this Proposition and Property 2 of 4.1.4, we may write

$$\begin{aligned} i(W) \circ (\mathcal{L}_\rho(V) \circ d_\rho - d_\rho \circ \mathcal{L}_\rho(V))\eta \\ &= \mathcal{L}_\rho(V) \circ i(W) \circ d_\rho \eta - i(\{V, W\}) \circ d_\rho \eta \\ &\quad - \mathcal{L}_\rho(W) \circ \mathcal{L}_\rho(V)\eta + d_\rho \circ i(W) \circ \mathcal{L}_\rho(V)\eta. \end{aligned}$$

By rearrangement of the terms, we obtain

$$\begin{aligned}
i(W) \circ (\mathcal{L}_\rho(V) \circ d_\rho - d_\rho \circ \mathcal{L}_\rho(V))\eta & \\
&= (\mathcal{L}_\rho(V) \circ \mathcal{L}_\rho(W) - \mathcal{L}_\rho(W) \circ \mathcal{L}_\rho(V) - \mathcal{L}_\rho(\{V, W\}))\eta \\
&\quad + d_\rho \circ i(\{V, W\})\eta - d_\rho \circ i(\{V, W\})\eta \\
&\quad - (\mathcal{L}_\rho(V) \circ d_\rho - d_\rho \circ \mathcal{L}_\rho(V))(i(W)\eta) \\
&= 0,
\end{aligned}$$

since $i(W)\eta \in \Omega^0(M, E)$, which implies that the last term vanishes.

4. Property 2 shows that d_ρ is a derivation of degree 1 of $\Omega(M, E)$. We know (3.1.6) that $[d_\rho, d_\rho] = 2d_\rho \circ d_\rho$ is a derivation of degree 2 of $\Omega(M, \mathbb{R})$. In order to prove that $d_\rho \circ d_\rho = 0$, it is enough to prove that it vanishes on $\Omega^0(M, E)$ and on $\Omega^1(M, E)$.

Let $f \in \Omega^0(M, E) = C^\infty(M, \mathbb{R})$, V and $W \in A^1(M, E)$. Then

$$\begin{aligned}
(d_\rho \circ d_\rho f)(V, W) &= \mathcal{L}_\rho(V)(d_\rho f(W)) - \mathcal{L}_\rho(W)(d_\rho f(V)) - d_\rho f(\{V, W\}) \\
&= \left(\mathcal{L}_\rho(V) \circ \mathcal{L}_\rho(W) - \mathcal{L}_\rho(W) \circ \mathcal{L}_\rho(V) - \mathcal{L}_\rho(\{V, W\}) \right) f \\
&= 0,
\end{aligned}$$

where we have used Property 4 of Proposition 4.1.4. We have shown that $d_\rho \circ d_\rho$ vanishes on $\Omega^0(M, E)$.

Now let $\eta \in \Omega^1(M, E)$, V_0, V_1 and $V_2 \in A^1(M, E)$. Using Property 1, we may write

$$\begin{aligned}
(d_\rho \circ d_\rho \eta)(V_0, V_1, V_2) &= \left((i(V_0) \circ d_\rho)(d_\rho \eta) \right)(V_1, V_2) \\
&= \left((\mathcal{L}_\rho(V_0) \circ d_\rho - d_\rho \circ i(V_0) \circ d_\rho) \eta \right)(V_1, V_2).
\end{aligned}$$

The last term in the right hand side may be transformed, by using again Property 1:

$$\begin{aligned}
d_\rho \circ i(V_0) \circ d_\rho(\eta) &= d_\rho \circ \mathcal{L}_\rho(V_0)\eta - d_\rho \circ d_\rho(i(V_0)\eta) \\
&= d_\rho \circ \mathcal{L}_\rho(V_0)\eta,
\end{aligned}$$

since, as $i(V_0)\eta \in \Omega^0(M, E)$, we have $d_\rho \circ d_\rho(i(V_0)\eta) = 0$. So we obtain

$$(d_\rho \circ d_\rho\eta)(V_0, V_1, V_2) = \left((\mathcal{L}_\rho(V_0) \circ d_\rho - d_\rho \circ \mathcal{L}_\rho(V_0))\eta \right)(V_1, V_2).$$

But Property 3 shows that

$$(\mathcal{L}_\rho(V_0) \circ d_\rho - d_\rho \circ \mathcal{L}_\rho(V_0))\eta = 0,$$

so we have

$$(d_\rho \circ d_\rho\eta)(V_0, V_1, V_2) = 0,$$

and our proof is complete. \times

4.3 Defining a Lie algebroid by properties of its dual

Let (E, π, M) be a vector bundle and (E^*, ϖ, M) its dual bundle. We have seen in 4.2 that when (E, π, M) has a Lie algebroid structure whose anchor is denoted by ρ , this structure determines, on the graded algebra $\Omega(M, E)$ of sections of the exterior powers of the dual bundle (E^*, ϖ, M) , a graded derivation d_ρ , of degree 1, which satisfies $d_\rho^2 = d_\rho \circ d_\rho = 0$. Now we are going to prove a converse of this property: when a graded derivation of degree 1, whose square vanishes, is given on $\Omega(M, E)$, it determines a Lie algebroid structure on (E, π, M) . This property will be used later to prove that the cotangent bundle of a Poisson manifold has a natural Lie algebroid structure.

We will need the following lemmas.

4.3.1 Lemma. *Let (E, π, M) be a vector bundle and (E^*, ϖ, M) its dual bundle. Let δ be a graded derivation of degree 1 of the exterior algebra $\Omega(M, E)$ (notations defined in 3.2.4). For each pair (X, Y) of smooth sections of (E, π, M) there exists a unique smooth section $[X, Y]_\delta$ of (E, π, M) , called the δ -bracket of X and Y , such that*

$$i([X, Y]_\delta) = [[i(X), \delta], i(Y)].$$

Proof:

The map defined by the right hand side of the above equality,

$$D : \eta \mapsto D(\eta) = [[i(X), \delta], i(Y)]$$

is a derivation of degree -1 of $\Omega(M, E)$, since it is obtained by repeated application of the graded bracket to derivations (property 3.1.6 (ii)). Therefore, it vanishes on $\Omega^0(M, E) = C^\infty(M, \mathbb{R})$. As a consequence, that map is $C^\infty(M, \mathbb{R})$ -linear; we have indeed, for each $f \in C^\infty(M, \mathbb{R})$ and $\eta \in \Omega(M, E)$,

$$D(f\eta) = D(f) \wedge \eta + fD(\eta) = fD(\eta).$$

Therefore, there exists a unique smooth section $[X, Y]_\delta$ of (E, π, M) such that, for each $\eta \in \Omega^1(M, E)$,

$$\langle \eta, [X, Y]_\delta \rangle = D(\eta).$$

Now the maps

$$i([X, Y]_\delta) \quad \text{and} \quad [[i(X), \delta], i(Y)]$$

are both derivations of degree -1 of $\Omega(M, E)$, which coincide on $\Omega^0(M, E)$ and $\Omega^1(M, E)$. Since derivations are local, and since $\Omega(M, E)$ is locally generated by its elements of degrees 0 and 1, these two derivations are equal. \bowtie

4.3.2 Lemma. *Under the same assumptions as those of Lemma 4.3.1, let us set, for each smooth section X of (E, π, M) ,*

$$\mathcal{L}_\delta(X) = [i(X), \delta].$$

Then, for each smooth section X of (E, π, M) , we have

$$[\mathcal{L}_\delta(X), \delta] = [i(X), \delta^2];$$

for each pair (X, Y) of smooth sections of (E, π, M) , we have

$$[\mathcal{L}_\delta(X), \mathcal{L}_\delta(Y)] - \mathcal{L}_\delta([X, Y]_\delta) = [[i(X), \delta^2], i(Y)]$$

and, for each triple (X, Y, Z) of smooth sections of (E, π, M) , we have

$$i\left([X, [Y, Z]_\delta]_\delta + [Y, [Z, X]_\delta]_\delta + [Z, [X, Y]_\delta]_\delta\right) = \left[[[i(X), \delta^2], i(Y)], i(Z)\right].$$

Proof:

Let us use the graded Jacobi identity. We may write

$$[\mathcal{L}_\delta(X), \delta] = [[i(X), \delta], \delta] = -[[\delta, \delta], i(X)] - [[\delta, i(X)], \delta].$$

Since $[\delta, \delta] = 2\delta^2$, we obtain

$$2[\mathcal{L}_\delta(X), \delta] = -2[\delta^2, i(X)] = 2[i(X), \delta^2],$$

which proves the first equality. Similarly, by using again the graded Jacobi identity and the equality just proven,

$$\begin{aligned} [\mathcal{L}_\delta(X), \mathcal{L}_\delta(Y)] &= [\mathcal{L}_\delta(X), [i(Y), \delta]] \\ &= -[i(Y), [\delta, \mathcal{L}_\delta(X)]] + [\delta, [\mathcal{L}_\delta(X), i(Y)]] \\ &= -[[\delta, \mathcal{L}_\delta(X)], i(Y)] + [[\mathcal{L}_\delta(X), i(Y)], \delta] \\ &= [[\mathcal{L}_\delta(X), \delta], i(Y)] + [i([X, Y]_\delta), \delta] \\ &= [[i(X), \delta^2], i(Y)] + \mathcal{L}_\delta([X, Y]_\delta). \end{aligned}$$

The second formula is proven. Finally,

$$\begin{aligned} i([X, [Y, Z]_\delta]_\delta) &= [\mathcal{L}_\delta(X), i([Y, Z]_\delta)] \\ &= [\mathcal{L}_\delta(X), [\mathcal{L}_\delta(Y), i(Z)]] \\ &= -[\mathcal{L}_\delta(Y), [i(Z), \mathcal{L}_\delta(X)]] - [i(Z), [\mathcal{L}_\delta(X), \mathcal{L}_\delta(Y)]] \\ &= [\mathcal{L}_\delta(Y), [\mathcal{L}_\delta(X), i(Z)]] - [i(Z), \mathcal{L}_\delta([X, Y]_\delta)] \\ &\quad - [i(Z), [[i(X), \delta^2], i(Y)]] \\ &= i([Y, [X, Z]_\delta]_\delta) + i([[X, Y]_\delta, Z]_\delta) \\ &\quad + [[i(X), \delta^2], i(Y), i(Z)]. \end{aligned}$$

The proof is complete. \bowtie

4.3.3 Theorem. *Let (E, π, M) be a vector bundle and (E^*, ϖ, M) its dual bundle. Let δ be a graded derivation of degree 1 of the exterior algebra $\Omega(M, E)$ (notations defined in 3.2.4), which satisfies*

$$\delta^2 = \delta \circ \delta = 0.$$

Then δ determines a natural Lie algebroid structure on (E, π, M) . For that structure, the anchor map $\rho : E \rightarrow TM$ is the unique vector bundle map such that, for each smooth section X of (E, π, M) and each function $f \in C^\infty(M, \mathbb{R})$,

$$i(\rho \circ X) df = \langle \delta f, X \rangle.$$

The bracket $(X, Y) \mapsto \{X, Y\}$ is the δ -bracket defined in Lemma 4.3.1; it is such that, for each pair (X, Y) of smooth sections of (E, π, M) ,

$$i(\{X, Y\}) = [[i(X), \delta], i(Y)].$$

The $\omega(M, E)$ -valued exterior derivative associated to that Lie algebroid structure (propositions 4.2.1 and 4.2.3) is the given derivation δ .

Proof:

Since $\delta^2 = 0$, lemmas 4.3.1 and 4.3.2 prove that the δ -bracket satisfies the Jacobi identity. Let X and Y be two smooth sections of (E, π, M) and f a smooth function on M . By using the definition of the δ -bracket we obtain

$$i([X, fY]_\delta) = f i([X, Y]_\delta) + (\mathcal{L}_\delta(X)f) i(Y).$$

But

$$\mathcal{L}_\delta(X)f = [i(X), \delta]f = \langle \delta f, X \rangle,$$

since $i(X)f = 0$. We must prove now that the value of $\delta(f)$ at any point $x \in M$ depends only on the value of the differential df of f at that point. We first observe that δ being a derivation, the values of $\delta(f)$ in some open subset U of M depend only on the values of f in that open subset. Moreover, we have

$$\delta(1f) = \delta(f) = \delta(1)f + 1\delta(f) = \delta(1)f + \delta(f),$$

which proves that δ vanishes on constants.

Let $a \in M$. We use a chart of M whose domain U contains a , and whose local coordinates are denoted by (x^1, \dots, x^n) . In order to calculate $\delta(f)(a)$, the above remarks allow us to work in that chart. We may write

$$f(x) = f(a) + \sum_{i=1}^n (x^i - a^i) \varphi_i(x), \quad \text{with} \quad \lim_{x \rightarrow a} \varphi_i(x) = \frac{\partial f}{\partial x^i}(a).$$

Therefore,

$$(\delta f)(a) = \sum_{i=1}^n \frac{\partial f}{\partial x^i}(a) \delta(x^i)(a).$$

We have proven that $\delta(f)(a)$ depends only on $df(a)$, and that we may write

$$\delta(f) = {}^t\rho_\delta \circ df,$$

where ${}^t\rho_\delta : T^*M \rightarrow E^*$ is a smooth vector bundle map. Let $\rho_\delta : E \rightarrow TM$ be its transpose. We may now write

$$[X, fY]_\delta = f[X, Y]_\delta + \langle df, \rho_\delta \circ X \rangle Y.$$

This proves that the vector bundle (E, π, M) , with the δ -bracket and the map ρ_δ as anchor, is a Lie algebroid. Finally, by using Propositions 4.2.1 and 4.2.2, we see that the $\Omega(M, E)$ -valued exterior derivative associated to that Lie algebroid structure is the derivation δ . \boxtimes

4.4 The Schouten-Nijenhuis bracket

In this subsection (E, π, M, ρ) is a Lie algebroid. We have seen (Propositions 4.1.4 and 4.1.6) that the composition law which associates, to each pair (V, W) of sections of the Lie algebroid (E, π, M, ρ) , the bracket $\{V, W\}$, extends into a map $(V, P) \mapsto \mathcal{L}_\rho(V)P$, defined on $A^1(M, E) \times A(M, E)$, with values in $A(M, E)$. Theorem 4.4.3 below will show that this map extends, in a very natural way, into a composition law $(P, Q) \mapsto [P, Q]$, defined on $A(M, E) \times A(M, E)$, with values in $A(M, E)$, called the *Schouten-Nijenhuis bracket*. That bracket was discovered by Schouten [24] for multivectors on a manifold, and its properties were further studied by Nijenhuis [22].

The following lemmas will be used in the proof of Theorem 4.4.3.

4.4.1 Lemma. *Let (E, π, M, ρ) be a Lie algebroid, p and $q \in \mathbb{Z}$, $P \in A^p(M, E)$,*

$Q \in A^q(M, E)$, $f \in C^\infty(M, \mathbb{R})$ and $\eta \in \Omega(M, E)$. Then

$$\begin{aligned} i(P)(df \wedge i(Q)\eta) &- (-1)^p df \wedge (i(P) \circ i(Q)\eta) \\ &+ (-1)^{(p-1)q} i(Q) \circ i(P)(df \wedge \eta) \\ &+ (-1)^{(p-1)q+p} i(Q)(df \wedge i(P)\eta) \\ &= 0. \end{aligned}$$

Proof:

Let us denote by $E(P, Q, f, \eta)$ the left hand side of the above equality. We have to prove that $E(P, Q, f, \eta) = 0$.

Obviously, $E(P, Q, f, \eta) = 0$ when $p < 0$, as well as when $q < 0$. When $p = q = 0$, we have

$$E(P, Q, f, \eta) = PQ df \wedge \eta - PQ df \wedge \eta - QP df \wedge \eta + QP df \wedge \eta = 0.$$

Now we proceed by induction on p and q , with the induction assumption that $E(P, Q, f, \eta) = 0$ when $p \leq p_M$ and $q \leq q_M$, for some integers p_M and q_M . Let $P = X \wedge P'$, with $X \in A^1(M, E)$ and $P' \in A^{p_M}(M, E)$, $Q \in A^q(M, E)$, with $q \leq q_M$, $f \in C^\infty(M, \mathbb{R})$ and $\eta \in \Omega(M, E)$. We obtain, after some calculations,

$$\begin{aligned} E(P, Q, f, \eta) &= E(X \wedge P', Q, f, \eta) \\ &= (-1)^{p_M+q-1} E(P', Q, f, i(X)\eta) \\ &\quad + (-1)^{p_M} \langle df, X \rangle i(P) \circ i(Q)\eta \\ &\quad - (-1)^{p_M+p_M q} \langle df, X \rangle i(Q) \circ i(P)\eta \\ &= 0, \end{aligned}$$

since, by the induction assumption, $E(P', Q, f, i(X)\eta) = 0$.

Since every $P \in A^{p_M+1}(M, E)$ is the sum of terms of the form $X \wedge P'$, with $X \in A^1(M, E)$ and $P' \in A^{p_M}(M, E)$, we see that $E(P, Q, f, \eta) = 0$ for all $p \leq p_M + 1$, $q \leq q_M$, $P \in A^{p_M+1}(M, E)$ and $Q \in A^{q_M}(M, E)$.

Moreover, P and Q play similar parts in $E(P, Q, f, \eta)$, since we have

$$E(P, Q, f, \eta) = (-1)^{pq+p+q} E(Q, P, f, \eta).$$

Therefore $E(P, Q, f, \eta) = 0$ for all $p \leq p_M + 1$, $q \leq q_M + 1$, $P \in A^p(M, E)$ and $Q \in A^q(M, E)$. By induction we conclude that $E(P, Q, f, \eta) = 0$ for all p and $q \in \mathbb{Z}$, $P \in A^p(M, E)$ and $Q \in A^q(M, E)$. \bowtie

4.4.2 Lemma. *Let (E, π, M, ρ) be a Lie algebroid, p, q and $r \in \mathbb{Z}$, $P \in A^p(M, E)$, $Q \in A^q(M, E)$ and $R \in A^r(M, E)$. Then*

$$i(R) \circ [[i(P), d_\rho], i(Q)] = (-1)^{(p+q-1)r} [[i(P), d_\rho], i(Q)] \circ i(R).$$

Proof:

Let us first assume that $R = V \in A^1(M, E)$. We may write

$$\begin{aligned} i(V) \circ [[i(P), d_\rho], i(Q)] &= i(V) \circ i(P) \circ d_\rho \circ i(Q) \\ &\quad - (-1)^p i(V) \circ d_\rho \circ i(P) \circ i(Q) \\ &\quad - (-1)^{(p-1)q} i(V) \circ i(Q) \circ i(P) \circ d_\rho \\ &\quad + (-1)^{(p-1)q+p} i(V) \circ i(Q) \circ d_\rho \circ i(P). \end{aligned}$$

We transform the right hand side by pushing the operator $i(V)$ towards the right, using the formulae (proven in 3.2.3 (v) and in Property 1 of 4.2.3)

$$i(V) \circ i(P) = (-1)^p i(P) \circ i(V) \quad \text{and} \quad i(V) \circ d_\rho = \mathcal{L}_\rho(V) - d_\rho \circ i(V).$$

We obtain, after rearrangement of the terms,

$$\begin{aligned} i(V) \circ [[i(P), d_\rho], i(Q)] &= (-1)^{p+q-1} [[i(P), d_\rho], i(Q)] \circ i(V) \\ &\quad + (-1)^p i(P) \circ \mathcal{L}_\rho(V) \circ i(Q) \\ &\quad - (-1)^p \mathcal{L}_\rho(V) \circ i(P) \circ i(Q) \\ &\quad - (-1)^{(p-1)q+p+q} i(Q) \circ i(P) \circ \mathcal{L}_\rho(V) \\ &\quad + (-1)^{(p-1)q+p+q} i(Q) \circ \mathcal{L}_\rho(V) \circ i(P). \end{aligned}$$

Now we transform the last four terms of the right hand side by pushing to the right the operator $\mathcal{L}_\rho(V)$, using formulae, proven in 3.2.3 (v) and in Property 4 of 4.1.6, of the type

$$i(P) \circ i(Q) = i(P \wedge Q) \quad \text{and} \quad \mathcal{L}_\rho(V) \circ i(P) = i(P) \circ \mathcal{L}_\rho(V) + i(\mathcal{L}_\rho(V)P).$$

The terms containing $\mathcal{L}_\rho(V)$ become

$$(-1)^p i \left(P \wedge \mathcal{L}_\rho(V) Q + (\mathcal{L}_\rho(V) P) \wedge Q - \mathcal{L}_\rho(V) (P \wedge Q) \right),$$

so they vanish, by Property 3 of 4.1.6. So we have

$$i(V) \circ [[i(P), d_\rho], i(Q)] = (-1)^{(p+q-1)} [[i(P), d_\rho], i(Q)] \circ i(V).$$

Now let $R = V_1 \wedge \cdots \wedge V_r$ be a decomposable element in $A^r(M, E)$. Since

$$i(R) = i(V_1) \circ \cdots \circ i(V_r),$$

by using r times the above result, we obtain

$$i(R) \circ [[i(P), d_\rho], i(Q)] = (-1)^{(p+q-1)r} [[i(P), d_\rho], i(Q)] \circ i(R).$$

Finally the same result holds for all $R \in A^r(M, E)$ by linearity. \bowtie

4.4.3 Theorem. *Let (E, π, M, ρ) be a Lie algebroid. Let p and $q \in \mathbb{Z}$, and let $P \in A^p(M, E)$, $Q \in A^q(M, E)$. There exists a unique element in $A^{p+q-1}(M, E)$, called the Schouten-Nijenhuis bracket of P and Q , and denoted by $[P, Q]$, such that the interior product $i([P, Q])$, considered as a graded endomorphism of degree $p + q - 1$ of the exterior algebra $\Omega(M, E)$, is given by the formula*

$$i([P, Q]) = [[i(P), d_\rho], i(Q)],$$

the brackets in the right hand side being the graded brackets of graded endomorphism (Definition 3.1.3).

Proof:

We observe that for all $r \in \mathbb{Z}$, the map

$$\eta \mapsto [[i(P), d_\rho], i(Q)] \eta,$$

defined on $\Omega^r(M, E)$, with values in $\Omega^{r-p-q+1}(M, E)$, is \mathbb{R} -linear. Let us prove that it is in fact $C^\infty(M, \mathbb{R})$ -linear. Let $f \in C^\infty(M, \mathbb{R})$. By developing the

double graded bracket of endomorphisms, we obtain after some calculations

$$\begin{aligned} [[i(P), d_\rho], i(Q)](f\eta) &= f[[i(P), d_\rho], i(Q)]\eta \\ &\quad + i(P)(df \wedge i(Q)\eta) - (-1)^p df \wedge (i(P) \circ i(Q)\eta) \\ &\quad + (-1)^{(p-1)q} i(Q) \circ i(P)(df \wedge \eta) \\ &\quad + (-1)^{(p-1)q+p} i(Q)(df \wedge i(P)\eta). \end{aligned}$$

Lemma 4.4.1 shows that the sum of the last four terms of the right hand side vanishes, so we obtain

$$[[i(P), d_\rho], i(Q)](f\eta) = f[[i(P), d_\rho], i(Q)]\eta.$$

Let us take $r = p + q - 1$, and $\eta \in \Omega^{p+q-1}(M, E)$. The map

$$\eta \mapsto [[i(P), d_\rho], i(Q)]\eta,$$

defined on $\Omega^{p+q-1}(M, E)$, takes its values in $\Omega^0(M, E) = C^\infty(M, \mathbb{R})$, and is $C^\infty(M, \mathbb{R})$ -linear. This proves the existence of a unique element $[P, Q]$ in $\Omega^{p+q-1}(M, E)$ such that, for all $\eta \in \Omega^{p+q-1}(M, E)$,

$$[[i(P), d_\rho], i(Q)]\eta = i([P, Q])\eta.$$

We still have to prove that the same formula holds for all $r \in \mathbb{Z}$ and all $\eta \in \Omega^r(M, E)$. The formula holds trivially when $r < p + q - 1$, so let us assume that $r > p + q - 1$. Let $\eta \in \Omega^r(M, E)$ and $R \in A^{r-p-q+1}(M, E)$. By using Lemma 4.4.2, we may write

$$\begin{aligned} i(R) \circ [[i(P), d_\rho], i(Q)](\eta) &= (-1)^{(p+q-1)(r-p-q+1)} [[i(P), d_\rho], i(Q)](i(R)\eta) \\ &= (-1)^{(p+q-1)(r-p-q+1)} i([P, Q])(i(R)\eta), \end{aligned}$$

since $i(R)\eta \in \Omega^{p+q-1}(E)$. Therefore

$$\begin{aligned} i(R) \circ [[i(P), d_\rho], i(Q)](\eta) &= (-1)^{(p+q-1)(r-p-q+1)} i([P, Q]) \circ i(R)\eta \\ &= i(R) \circ i([P, Q])\eta. \end{aligned}$$

Since that equality holds for all $\eta \in \Omega^r(M, E)$ and all $R \in A^{r-p-q+1}(M, E)$, we may conclude that

$$[[i(P), d_\rho], i(Q)] = i([P, Q]),$$

and the proof is complete. \bowtie

In Proposition 4.1.1, we introduced the Lie derivative with respect to a section of the Lie algebroid (E, π, M, ρ) . Now we define, for all $p \in \mathbb{Z}$ and $P \in A^p(M, E)$, the Lie derivative with respect to P . The reader will observe that Property 1 of Proposition 4.2.3 shows that for $p = 1$, the following definition is in agreement with the definition of the Lie derivative with respect to an element in $A^1(M, E)$ given in 4.1.1.

4.4.4 Definition. *Let (E, π, M, ρ) be a Lie algebroid, $p \in \mathbb{Z}$ and $P \in A^p(M, E)$. The Lie derivative with respect to P is the graded endomorphism of $\Omega(M, P)$, of degree $1 - p$, denoted by $\mathcal{L}_\rho(P)$,*

$$\mathcal{L}_\rho(P) = [i(P), d_\rho] = i(P) \circ d_\rho - (-1)^p d_\rho \circ i(P).$$

4.4.5 Remark. Under the assumptions of Theorem 4.4.3, the above Definition allows us to write

$$i([P, Q]) = [\mathcal{L}_\rho(P), i(Q)] = \mathcal{L}_\rho(P) \circ i(Q) - (-1)^{(p-1)q} i(Q) \circ \mathcal{L}_\rho(P).$$

For $p = 1$ and $P = V \in A^1(M, E)$, this formula is simply Property 4 of Proposition 4.1.6, as shown by the following Proposition.

4.4.6 Proposition. *Under the assumptions of Theorem 4.4.3, let $p = 1$, $P = V \in A^1(M, E)$ and $Q \in A^q(M, E)$. The Schouten-Nijenhuis bracket $[V, Q]$ is simply the Lie derivative of Q with respect to V , as defined in Proposition 4.1.5:*

$$[V, Q] = \mathcal{L}_\rho(V)Q.$$

Proof:

As seen in Remark 4.4.5, we may write

$$i([V, Q]) = [\mathcal{L}_\rho(V), i(Q)] = \mathcal{L}_\rho(V) \circ i(Q) - i(Q) \circ \mathcal{L}_\rho(V).$$

Property 4 of Proposition 4.1.6 shows that

$$i(\mathcal{L}_\rho(V)Q) = \mathcal{L}_\rho(V) \circ i(Q) - i(Q) \circ \mathcal{L}_\rho(V).$$

Therefore,

$$i([V, Q]) = i(\mathcal{L}_\rho(V)Q),$$

and finally

$$[V, Q] = \mathcal{L}_\rho(V)Q,$$

which ends the proof. \square

4.4.7 Remarks.

(i) *The Lie derivative of elements in $A^p(M, E)$.* One may think to extend the range of application of the Lie derivative with respect to a multivector $P \in A^p(M, E)$ by setting, for all $q \in \mathbb{Z}$ and $Q \in A^q(M, E)$,

$$\mathcal{L}_\rho(P)Q = [P, Q],$$

the bracket in the right hand side being the Schouten-Nijenhuis bracket. However, we will avoid the use of that notation because it may lead to confusions: for $p > 1$, $P \in A^p(M, E)$, $q = 0$ and $Q = f \in A^0(M, E) = C^\infty(M, \mathbb{R})$, the Schouten-Nijenhuis bracket $[P, f]$ is an element in $A^{p-1}(M, E)$ which does not vanish in general. But f can be considered also as an element in $\Omega^0(M, E)$, and the Lie derivative of f with respect to P , in the sense of Definition 4.4.4, is an element in $\Omega^{-(p-1)}(M, E)$, therefore vanishes identically. So it would not be a good idea to write $\mathcal{L}_\rho(P)f = [P, f]$.

(ii) *Lie derivatives and derivations.* We have seen (Property 3 of 4.1.4) that the Lie derivative $\mathcal{L}_\rho(V)$ with respect to a section $V \in A^1(M, R)$ of the Lie algebroid (E, π, M, ρ) is a derivation of degree 0 of the exterior algebra $\Omega(M, E)$. For $p > 1$ and $P \in A^p(M, E)$, the Lie derivative $\mathcal{L}_\rho(P)$ is a graded endomorphism of degree $-(p-1)$ of $\Omega(M, E)$. Therefore, it vanishes identically on $\Omega^0(M, E)$ and on $\Omega^1(M, E)$. Unless it vanishes identically, $\mathcal{L}_\rho(P)$ is not a derivation of $\Omega(M, E)$.

4.4.8 Proposition. *Let (E, π, M, ρ) be a Lie algebroid, p and $q \in \mathbb{Z}$, $P \in A^p(M, E)$, $Q \in A^q(M, E)$.*

1. *The graded bracket of the Lie derivative $\mathcal{L}_\rho(P)$ and the exterior differen-*

tial d_ρ vanishes identically:

$$[\mathcal{L}_\rho(P), d_\rho] = \mathcal{L}_\rho(P) \circ d_\rho - (-1)^{p-1} d_\rho \circ \mathcal{L}_\rho(P) = 0.$$

2. The graded bracket of the Lie derivatives $\mathcal{L}_\rho(P)$ and $\mathcal{L}_\rho(Q)$ is equal to the Lie derivative $\mathcal{L}_\rho([P, Q])$:

$$[\mathcal{L}_\rho(P), \mathcal{L}_\rho(Q)] = \mathcal{L}_\rho(P) \circ \mathcal{L}_\rho(Q) - (-1)^{(p-1)(q-1)} \mathcal{L}_\rho(Q) \circ \mathcal{L}_\rho(P) = \mathcal{L}_\rho([P, Q]).$$

Proof:

1. We have seen (3.1.8 (ii)) that the space of graded endomorphisms of $\Omega(M, E)$, with the graded bracket as composition law, is a graded Lie algebra. By using the graded Jacobi identity, we may write

$$(-1)^p [[i(P), d_\rho], d_\rho] + (-1)^p [[d_\rho, d_\rho], i(P)] - [[d_\rho, i(P)], d_\rho] = 0.$$

But

$$[d_\rho, d_\rho] = 2d_\rho \circ d_\rho = 0 \quad \text{and} \quad [i(P), d_\rho] = -(-1)^p [d_\rho, i(P)].$$

So we obtain

$$2[[i(P), d_\rho], d_\rho] = 2[\mathcal{L}_\rho(P), d_\rho] = 0.$$

2. We have

$$\mathcal{L}_\rho([P, Q]) = [i([P, Q]), d_\rho] = [[\mathcal{L}_\rho(P), i(Q)], d_\rho].$$

Using the graded Jacobi identity, we may write

$$\begin{aligned} (-1)^{p-1} [[\mathcal{L}_\rho(P), i(Q)], d_\rho] + (-1)^{q(p-1)} [[i(Q), d_\rho], \mathcal{L}_\rho(P)] \\ + (-1)^q [[d_\rho, \mathcal{L}_\rho(P)], i(Q)] = 0. \end{aligned}$$

But, according to 4.4.4 and Property 1 above,

$$[i(Q), d_\rho] = \mathcal{L}_\rho(Q) \quad \text{and} \quad [d_\rho, \mathcal{L}_\rho(P)] = 0.$$

So we obtain

$$\mathcal{L}_\rho([P, Q]) = -(-1)^{(p-1)(q-1)} [\mathcal{L}_\rho(Q), \mathcal{L}_\rho(P)] = [\mathcal{L}_\rho(P), \mathcal{L}_\rho(Q)],$$

as announced. \bowtie

4.4.9 Proposition. *Under the same assumptions as those of Theorem 4.4.3, the Schouten-Nijenhuis bracket has the following properties.*

1. For f and $g \in A^0(M, E) = C^\infty(M, \mathbb{R})$,

$$[f, g] = 0.$$

2. For $V \in A^1(M, E)$, $q \in \mathbb{Z}$ and $Q \in A^q(M, E)$,

$$[V, Q] = \mathcal{L}_\rho(V)Q.$$

3. For V and $W \in A^1(M, E)$,

$$[V, W] = \{V, W\},$$

the bracket in the right hand side being the Lie algebroid bracket.

4. For all p and $q \in \mathbb{Z}$, $P \in A^p(M, E)$, $Q \in A^q(M, E)$,

$$[P, Q] = -(-1)^{(p-1)(q-1)}[Q, P].$$

5. Let $p \in \mathbb{Z}$, $P \in A^p(M, E)$. The map $Q \mapsto [P, Q]$ is a derivation of degree $p-1$ of the graded exterior algebra $A(M, E)$. In other words, for q_1 and $q_2 \in \mathbb{Z}$, $Q_1 \in A^{q_1}(M, E)$ and $Q_2 \in A^{q_2}(M, E)$,

$$[P, Q_1 \wedge Q_2] = [P, Q_1] \wedge Q_2 + (-1)^{(p-1)q_1} Q_1 \wedge [P, Q_2].$$

6. Let p , q and $r \in \mathbb{Z}$, $P \in A^p(M, E)$, $Q \in A^q(M, E)$ and $R \in A^r(M, E)$.

The Schouten-Nijenhuis bracket satisfies the graded Jacobi identity:

$$\begin{aligned} & (-1)^{(p-1)(r-1)} [[P, Q], R] + (-1)^{(q-1)(p-1)} [[Q, R], P] \\ & \quad + (-1)^{(r-1)(q-1)} [[R, P], Q] \\ & = 0. \end{aligned}$$

Proof:

1. Let f and $g \in A^0(M, E)$. Then $[f, g] \in A^{-1}(M, E) = \{0\}$, therefore $[f, g] = 0$.

2. See Proposition 4.4.6.

3. See Property 1 of Proposition 4.1.6.

4. Let p and $q \in \mathbb{Z}$, $P \in A^p(M, E)$, $Q \in A^q(M, E)$. By using the graded Jacobi identity for graded endomorphisms of $\Omega(M, E)$, we may write

$$(-1)^{pq} [[i(P), d_\rho], i(Q)] + (-1)^p [[d_\rho, i(Q)], i(P)] + (-1)^q [[i(Q), i(P)], d_\rho] = 0.$$

By using

$$[i(Q), i(P)] = i(Q \wedge P) - i(Q \wedge P) = 0 \quad \text{and} \quad [d_\rho, i(Q)] = -(-1)^q [i(Q), d_\rho],$$

we obtain

$$(-1)^{pq} [[i(P), d_\rho], i(Q)] + (-1)^{p+q-1} [[i(Q), d_\rho], i(P)] = 0,$$

so the result follows immediately.

5. Let p, q_1 and $q_2 \in \mathbb{Z}$, $P \in A^p(M, E)$, $Q_1 \in A^{q_1}(M, E)$ and $Q_2 \in A^{q_2}(M, E)$. We may write

$$\begin{aligned} i([P, Q_1 \wedge Q_2]) &= [\mathcal{L}_\rho(P), i(Q_1 \wedge Q_2)] \\ &= \mathcal{L}_\rho(P) \circ i(Q_1 \wedge Q_2) \\ &\quad - (-1)^{(p-1)(q_1+q_2)} i(Q_1 \wedge Q_2) \circ \mathcal{L}_\rho(P). \end{aligned}$$

We add and subtract $(-1)^{(p-1)q_1} i(Q_1) \circ \mathcal{L}_\rho(P) \circ i(Q_1)$ from the last expression, and replace $i(Q_1 \wedge Q_2)$ by $i(Q_1) \circ i(Q_2)$. We obtain

$$i([P, Q_1 \wedge Q_2]) = [\mathcal{L}_\rho(P), i(Q_1)] \circ i(Q_2) + (-1)^{(p-1)q_1} i(Q_1) \circ [\mathcal{L}_\rho(P), i(Q_2)].$$

The result follows immediately.

6. Let p, q and $r \in \mathbb{Z}$, $P \in A^p(M, E)$, $Q \in A^q(M, E)$ and $R \in A^r(M, E)$. By using Property 2 of Proposition 4.4.8, we may write

$$\begin{aligned} i([P, Q], R) &= [\mathcal{L}_\rho([P, Q]), i(R)] \\ &= [[\mathcal{L}_\rho(P), \mathcal{L}_\rho(Q)], i(R)]. \end{aligned}$$

Using the graded Jacobi identity, we obtain

$$\begin{aligned} (-1)^{(p-1)r} [[\mathcal{L}_\rho(P), \mathcal{L}_\rho(Q)], i(R)] + (-1)^{(q-1)(p-1)} [[\mathcal{L}_\rho(Q), i(R)], \mathcal{L}_\rho(P)] \\ + (-1)^{r(q-1)} [[i(R), \mathcal{L}_\rho(P)], \mathcal{L}_\rho(Q)] = 0. \end{aligned}$$

But

$$\begin{aligned} [[\mathcal{L}_\rho(Q), i(R)], \mathcal{L}_\rho(P)] &= [i([Q, R]), \mathcal{L}_\rho(P)] \\ &= -(-1)^{(q+r-1)(p-1)} [\mathcal{L}_\rho(P), i([Q, R])] \\ &= -(-1)^{(q+r-1)(p-1)} i([P, [Q, R]]) \\ &= (-1)^{(q+r-1)(p-1)+(p-1)(q+r-2)} i([[Q, R], P]) \\ &= (-1)^{p-1} i([[Q, R], P]). \end{aligned}$$

Similarly,

$$\begin{aligned} [[i(R), \mathcal{L}_\rho(P)], \mathcal{L}_\rho(Q)] &= -(-1)^{(p-1)r} [[\mathcal{L}_\rho(P), i(R)], \mathcal{L}_\rho(Q)] \\ &= -(-1)^{(p-1)r} [i([P, R]), \mathcal{L}_\rho(Q)] \\ &= (-1)^{(p-1)r+(p+r-1)(q-1)} [\mathcal{L}_\rho(Q), i([P, R])] \\ &= (-1)^{(p-1)(r+q-1)+r(q-1)} i([Q, [P, R]]) \\ &= -(-1)^{(p-1)q+(q-1)(p-2)} i([[R, P], Q]) \\ &= -(-1)^{p+q} i([[R, P], Q]). \end{aligned}$$

Using the above equalities, we obtain

$$\begin{aligned} (-1)^{(p-1)(r-1)} i([[P, Q]), R)] + (-1)^{(q-1)(p-1)} i([[Q, R], P]) \\ + (-1)^{(r-1)(q-1)} i([[R, P], Q]) \\ = 0. \end{aligned}$$

The proof is complete. \boxtimes

4.4.10 Remarks. Let (E, π, M, ρ) be a Lie algebroid.

(i) *Degrees for the two algebra structures of $A(M, E)$.* The exterior algebra $A(M, E) = \bigoplus_{p \in \mathbb{Z}} A^p(M, E)$ of sections of the exterior powers $(\bigwedge^p E, \pi, M)$, with the exterior product as composition law, is a graded associative algebra; for

that structure, the space of homogeneous elements of degree p is $A^p(M, E)$. Proposition 4.4.9 shows that $A(M, E)$, with the Schouten-Nijenhuis bracket as composition law, is a graded Lie algebra; for that structure, the space of homogeneous elements of degree p is not $A^p(M, E)$, but rather $A^{p+1}(M, E)$. For homogeneous elements in $A(M, E)$, one should therefore make a distinction between the degree for the graded associative algebra structure and the degree for the graded Lie algebra structure; an element in $A^p(M, E)$ has degree p for the graded associative algebra structure, and degree $p - 1$ for the graded Lie algebra structure.

(ii) *The anchor as a graded Lie algebras homomorphism.* The anchor $\rho : E \rightarrow TM$ allows us to associate to each smooth section $X \in A^1(M, E)$ a smooth vector field $\rho \circ X$ on M ; according to Definition 2.1.1, that correspondence is a Lie algebras homomorphism. We can extend that map, for all $p \geq 1$, to the space $A^p(M, E)$ of smooth sections of the p -th external power $(\wedge^p E, \pi, M)$. First, for a decomposable element $X_1 \wedge \cdots \wedge X_p$, with $X_i \in A^1(M, E)$, we set

$$\rho \circ (X_1 \wedge \cdots \wedge X_p) = (\rho \circ X_1) \wedge \cdots \wedge (\rho \circ X_p).$$

For $p = 0$, $f \in A^0(M, E) = C^\infty(M, \mathbb{R})$, we set, as a convention,

$$\rho \circ f = f.$$

Then we extend that correspondence to all elements in $A(M, E)$ by $C^\infty(M, \mathbb{R})$ -linearity. The map $P \mapsto \rho \circ P$ obtained in that way is a homomorphism from $A(M, E)$ into $A(M, TM)$, both for their graded associative algebras structures (with the exterior products as composition laws) and their graded Lie algebras structures (with the Schouten-Nijenhuis brackets, associated to the Lie algebroid structure of (E, π, M, ρ) and to the Lie algebroid structure of the tangent bundle $(TM, \tau_M, M, \text{id}_{TM})$ as composition laws).

In 5.2.2 (iii), we will see that when the Lie algebroid under consideration is the cotangent bundle to a Poisson manifold, the anchor map has still an additional property: it induces a cohomology anti-homomorphism.

5 Poisson manifolds and Lie algebroids

In this final section we will show that there exist very close links between Poisson manifolds and Lie algebroids.

5.1 Poisson manifolds

Poisson manifolds were introduced by A. Lichnerowicz in the very important paper [17]. Their importance was soon recognized, and their properties were investigated in depth by A. Weinstein [26]. Let us recall briefly their definition and some of their properties. The reader is referred to [17, 26, 25] for the proofs of these properties.

5.1.1 Definition. *Let M be a smooth manifold. We assume that the space $C^\infty(M, \mathbb{R})$ of smooth functions on M is endowed with a composition law, denoted by $(f, g) \mapsto \{f, g\}$, for which $C^\infty(M, \mathbb{R})$ is a Lie algebra, which moreover satisfies the Leibniz-type formula*

$$\{f, gh\} = \{f, g\}h + g\{f, h\}.$$

We say that the structure defined on M by such a composition law is a Poisson structure, and that the manifold M , equipped with that structure, is a Poisson manifold.

The following Proposition is due to A. Lichnerowicz [17]. Independently, A. Kirillov [10] introduced local Lie algebras (which include both Poisson manifolds and Jacobi manifolds, which were introduced too by A. Lichnerowicz [18]) and obtained, without using the Schouten-Nijenhuis bracket, an equivalent result and its generalization for Jacobi manifolds.

5.1.2 Proposition. *On a Poisson manifold M , there exists a unique smooth section $\Lambda \in A^2(M, TM)$, called the Poisson bivector, which satisfies*

$$[\Lambda, \Lambda] = 0, \tag{*}$$

such that for any f and $g \in C^\infty(M, \mathbb{R})$,

$$\{f, g\} = \Lambda(df, dg). \quad (**)$$

The bracket in the left hand side of (*) is the Schouten-Nijenhuis bracket of multivectors on M , for the canonical Lie algebroid structure of (TM, τ_M, M) (with id_{TM} as anchor).

Conversely, let Λ be a smooth section of $A^2(TM, M)$. We use formula (**) to define a composition law on $C^\infty(M, \mathbb{R})$. The structure defined on M by that composition law is a Poisson structure if and only if Λ satisfies formula (*).

In what follows, we will denote by (M, Λ) a manifold M equipped with a Poisson structure whose Poisson bivector is Λ .

5.2 The Lie algebroid structure on the cotangent bundle of a Poisson manifold

The next theorem shows that the cotangent bundle of a Poisson manifold has a canonical structure of Lie algebroid. That property was discovered by Dazord and Sondaz [5].

5.2.1 Theorem. *Let (M, Λ) be a Poisson manifold. There exists, on the cotangent bundle (T^*M, π_M, M) , a canonical structure of Lie algebroid characterized by the following properties:*

- the bracket $[\eta, \zeta]$ of two sections η and ζ of (T^*M, π_M, M) , i.e., of two Pfaff forms on M , is given by the formula

$$\langle [\eta, \zeta], X \rangle = \langle \eta, [\Lambda, \langle \zeta, X \rangle] \rangle - \langle \zeta, [\Lambda, \langle \eta, X \rangle] \rangle - [\Lambda, X](\eta, \zeta),$$

where X is any smooth vector field on M ; the bracket in the right hand side of that formula is the Schouten-Nijenhuis bracket of multivectors on M ;

- the anchor is the vector bundle map $\Lambda^\sharp : T^*M \rightarrow TM$ such that, for each $x \in M$, α and $\beta \in T_x^*M$,

$$\langle \beta, \Lambda^\sharp \alpha \rangle = \Lambda(\alpha, \beta) , .$$

Proof:

We define a linear endomorphism δ_Λ of $A(M, TM)$ by setting, for each $P \in A(M, TM)$,

$$\delta_\Lambda(P) = [\Lambda, P] ,$$

where the bracket in the right hand side is the Schouten-Nijenhuis bracket of multivectors on M , *i.e.*, the Schouten-Nijenhuis bracket for the canonical Lie algebroid structure of (TM, τ_M, M) (with id_{TM} as anchor map). When P is in $A^p(M, TM)$, $\delta_\Lambda(P)$ is in $A^{p+1}(M, TM)$, therefore δ_Λ is homogeneous of degree 1. For each $P \in A^p(M, TM)$ and $Q \in A^q(M, TM)$, we have

$$\begin{aligned} \delta_\Lambda(P \wedge Q) &= [\Lambda, P \wedge Q] \\ &= [\Lambda, P] \wedge Q + (-1)^p P \wedge [\Lambda, Q] \\ &= \delta_\Lambda(P) \wedge Q + P \wedge \delta_\Lambda(Q) . \end{aligned}$$

This proves that δ_Λ is a graded derivation of degree 1 of the exterior algebra $A(M, TM)$.

Moreover, for each $P \in A^p(M, TM)$ we obtain, by using the graded Jacobi identity,

$$\begin{aligned} \delta_\Lambda \circ \delta_\Lambda(P) &= [\Lambda, [\Lambda, P]] \\ &= (-1)^{p-1} [\Lambda, [P, \Lambda]] - [P, [\Lambda, \Lambda]] \\ &= -[\Lambda, [\Lambda, P]] - [P, [\Lambda, \Lambda]] \\ &= -\delta_\Lambda \circ \delta_\Lambda(P) - [P, [\Lambda, \Lambda]] . \end{aligned}$$

Therefore

$$2\delta_\Lambda \circ \delta_\Lambda(P) = -[P, [\Lambda, \Lambda]] = 0 ,$$

since $[\Lambda, \Lambda] = 0$. We have proven that the graded derivation δ_Λ , of degree 1, satisfies

$$\delta_\Lambda^2 = \delta_\Lambda \circ \delta_\Lambda = 0.$$

Now we observe that the tangent bundle (TM, τ_M, M) can be considered as the dual bundle of the cotangent bundle (T^*M, π_M, M) . Therefore, we may apply Theorem 4.3.3, which shows that there exists on (T^*M, π_M, M) a Lie algebroid structure for which δ_M is the associated derivation on the space $\Omega(M, T^*M) = A(M, TM)$ (with the notations defined in 3.2.4). That theorem also shows that the bracket of two smooth sections of (T^*M, π_M, M) , *i.e.*, of two Pfaff forms η and ζ on M , is given by the formula, where X is any smooth vector field on M ,

$$\langle [\eta, \zeta], X \rangle = \langle \eta, [\Lambda, \langle \zeta, X \rangle] \rangle - \langle \zeta, [\Lambda, \langle \eta, X \rangle] \rangle - [\Lambda, X](\eta, \zeta).$$

The anchor map ρ is such that, for each $\eta \in \Omega^1(M, TM)$ and each $f \in C^\infty(M, \mathbb{R})$,

$$i(\rho \circ \eta) df = \langle \eta, [\Lambda, f] \rangle.$$

The bracket which appears in the right hand sides of these two formulae is the Schouten-Nijenhuis bracket of multivectors on M . By using Theorem 4.4.3, we see that

$$[\Lambda, f] = -\Lambda^\sharp(df).$$

Therefore,

$$\langle df, \rho \circ \eta \rangle = i(\rho \circ \eta) df = \langle \eta, -\Lambda^\sharp(df) \rangle = \langle df, \Lambda^\sharp(\eta) \rangle.$$

So we have $\rho = \Lambda^\sharp$. \(\times\)

5.2.2 Remarks. Let (M, Λ) be a Poisson manifold.

(i) *The bracket of forms of any degrees on M .* Since, by Theorem 5.2.1, $(T^*M, \pi_M, M, \Lambda^\sharp)$ is a Lie algebroid, we can define a composition law in the space $A(M, T^*M) = \Omega(M, \mathbb{R})$ of smooth differential forms of all degrees on M : the Schouten-Nijenhuis bracket for the Lie algebroid structure of (T^*M, π_M, M) ,

with Λ^\sharp as anchor. With that composition law, denoted by $(\eta, \zeta) \mapsto [\eta, \zeta]$, $\Omega(M, \mathbb{R})$ is a graded Lie algebra. Observe that a form $\eta \in \Omega^p(M, \mathbb{R})$, of degree p for the graded associative algebra structure whose composition law is the exterior product, has degree $p - 1$ for the graded Lie algebra structure.

The bracket of differential forms on a Poisson manifold was first discovered for Pfaff forms by Magri and Morosi [21]. It is related to the Poisson bracket of functions by the formula

$$[df, dg] = d\{f, g\}, \quad \text{with } f \text{ and } g \in C^\infty(M, \mathbb{R}).$$

That bracket was extended to forms of all degrees by Koszul [13], and rediscovered, with the Lie algebroid structure of T^*M , by Dazord and Sondaz [5].

(ii) *The Lichnerowicz-Poisson cohomology.* The derivation δ_Λ ,

$$P \mapsto \delta_\Lambda(P) = [\Lambda, P], \quad P \in A(M, TM),$$

used in the proof of Theorem 5.2.1, was first introduced by A. Lichnerowicz [17], who observed that it may be used to define a cohomology with elements in $A(M, TM)$ as cochains. He began the study of that cohomology, often called the Poisson cohomology (but which should be called the Lichnerowicz-Poisson cohomology). The study of that cohomology was carried on by Vaisman [25], Huebschmann [7], Xu [29] and many other authors.

(iii) *The map Λ^\sharp as a cohomology anti-homomorphism.* In 4.4.10 (ii), we have seen that the anchor map ρ of a Lie algebroid (E, π, M, ρ) yields a map $P \mapsto \rho \circ P$ from $A(M, E)$ into $A(M, TM)$, which is both a homomorphism of graded associative algebras (the composition laws being the exterior products) and a homomorphism of graded Lie algebras (the composition laws being the Schouten brackets). When applied to the Lie algebroid $(T^*M, \pi_M, M, \Lambda^\sharp)$, that property shows that the map $\eta \mapsto \Lambda^\sharp \circ \eta$ is a homomorphism from the space of differential forms $\Omega(M, \mathbb{R})$ into the space of multivectors $A(M, \mathbb{R})$, both for their structures of graded associative algebras and their structures of graded Lie algebras. As observed by A. Lichnerowicz [17], that map exchanges the exterior derivation d

of differential forms and the derivation δ_Λ of multivectors (with a sign change, under our sign conventions), in the following sense: for any $\eta \in \Omega^p(M, \mathbb{R})$, we have

$$\Lambda^\#(d\eta) = -\delta_\Lambda(\Lambda^\#(\eta)) = -[\Lambda, \Lambda^\#(\eta)].$$

That property is an easy consequence of the formula, valid for any smooth function $f \in C^\infty(M, \mathbb{R})$, which can be derived from Theorem 4.4.3,

$$\Lambda^\#(df) = -[\Lambda, f].$$

The map $\Lambda^\#$ therefore induces an anti-homomorphism from the Lichnerowicz-Poisson cohomology of the Poisson manifold (M, Λ) , into its De Rham cohomology.

(iv) *Lie bialgebroids.* Given a Poisson manifold (M, Λ) , we have Lie algebroid structures both on the tangent bundle (TM, τ_M, M) and on the cotangent bundle (T^*M, π_M, M) , with $\text{id}_{TM} : TM \rightarrow TM$ and $\Lambda^\# : T^*M \rightarrow TM$ as their respective anchor maps. Moreover, these two Lie algebroid structures are compatible in the following sense: the derivation $\delta_\Lambda : P \mapsto [\Lambda, P]$ of the graded associative algebra $A(M, TM)$ (the composition law being the exterior product) determined by the Lie algebroid structure of (T^*M, π_M, M) is also a derivation for the graded Lie algebra structure of $A(M, E)$ (the composition law being now the Schouten-Nijenhuis bracket). We have indeed, as an easy consequence of the graded Jacobi identity, for $P \in A^p(M, TM)$ and $Q \in A^q(M, TM)$,

$$\begin{aligned} \delta_\Lambda([P, Q]) &= [\Lambda, [P, Q]] = [[\Lambda, P], Q] + (-1)^{p-1}[P, [\Lambda, Q]] \\ &= [\delta_\Lambda P, Q] + (-1)^{p-1}[P, \delta_\Lambda Q]. \end{aligned}$$

When two Lie algebroid structures on two vector bundles in duality satisfy such a compatibility condition, it is said that that pair of Lie algebroids is a *Lie bialgebroid*. The very important notion of a Lie bialgebroid is due to K. Mackenzie and P. Xu [20]. Its study was developed by Y. Kosmann-Schwarzbach [11] and her student [2] and many other authors. Recently D. Iglesias and J.C. Marrero have introduced a generalization of that notion in relation with Jacobi manifolds [8].

5.3 The Poisson structure on the dual bundle of a Lie algebroid

We will now prove that the total space of the dual bundle $(E^*\varpi, M)$ of a Lie algebroid (E, π, M, ρ) has a canonical Poisson structure. This result will allow us to recover well known results:

- by taking as Lie algebroid the tangent bundle $(TM, \tau_M, M, \text{id}_{TM})$ of a smooth manifold M , we obtain on the cotangent bundle T^*M a Poisson structure, which is the structure associated to its canonical symplectic structure;
- by taking as Lie algebroid a finite-dimensional Lie algebra \mathfrak{G} (the base M being here reduced to a point) we recover, on its dual \mathfrak{G}^* , the canonical Lie-Poisson structure.

We will need the following lemma.

5.3.1 Lemma. *Let (E, π, M, ρ) be a Lie algebroid and (E^*, ϖ, M) its dual bundle. To each smooth section $X \in A^1(M, E)$, we associate a smooth function Φ_X on E^* by setting*

$$\Phi_X(\xi) = \langle \xi, X \circ \varpi(\xi) \rangle.$$

Let $X \in A^1(M, E)$, $f \in C^\infty(M, \mathbb{R})$ and $\xi \in E^*$ be such that

$$d(\Phi_X + f \circ \varpi)(\xi) = 0.$$

Then we have

$$X \circ \varpi(\xi) = 0$$

and, for any smooth section $Y \in A^1(M, E)$,

$$\Phi_{\{X, Y\}}(\xi) = \langle df, \rho \circ Y \rangle \circ \varpi(\xi).$$

Proof:

Since $d(\Phi_X + f \circ \varpi)(\xi) = 0$ we have, for any vector $W \in T_\xi E^*$,

$$\langle d\Phi_X(\xi), W \rangle = -\langle df(\varpi(\xi)), T_\xi \varpi(W) \rangle. \quad (*)$$

Let us first assume that W is vertical, *i.e.*, that $T_\xi \varpi(W) = 0$. Then W can be identified with an element of $\varpi^{-1}(\varpi(\xi))$, and $\varpi(\xi + sW) = \varpi(\xi)$ for all $s \in \mathbb{R}$. Therefore,

$$\begin{aligned} \langle d\Phi_X(\xi), W \rangle &= \frac{d}{ds} \Phi_X(\xi + sW) \Big|_{s=0} \\ &= \frac{d}{ds} \langle \xi + sW, X \circ \varpi(\xi + sW) \rangle \Big|_{s=0} \\ &= \langle W, X \circ \varpi(\xi) \rangle \\ &= 0. \end{aligned}$$

Since that last equality holds for all $W \in \varpi^{-1}(\varpi(\xi))$, we have $X \circ \varpi(\xi) = 0$.

Let (s_1, \dots, s_k) be a family of smooth sections of π defined on an open subset $U \subset M$, with $\varpi(\xi) \in U$, such that, for each $y \in U$, $(s_1(y), \dots, s_k(y))$ is a basis of the fibre $\pi^{-1}(y)$. We have on U

$$X = \sum_{i=1}^k X^i s_i,$$

where the X^i are smooth functions on U which satisfy $X^i \circ \varpi(\xi) = 0$. Then we have, for all $\eta \in \varpi^{-1}(U)$,

$$\Phi_X(\eta) = \sum_{i=1}^k X^i \circ \varpi(\eta) \langle \eta, s_i \circ \varpi(\eta) \rangle.$$

Let us now consider a vector $W \in T_\xi E^*$, which may not be vertical, and a smooth curve $s \mapsto \eta(s)$ in E^* such that $\eta(0) = \xi$ and $\frac{d}{ds} \eta(s) \Big|_{s=0} = W$. We have

$$\begin{aligned} \langle d\Phi_X(\xi), W \rangle &= \frac{d}{ds} \Phi_X(\eta(s)) \Big|_{s=0} \\ &= \sum_{i=1}^k \frac{d}{ds} (X^i \circ \varpi(\eta(s))) \Big|_{s=0} \langle \xi, s_i \circ \varpi(\xi) \rangle \\ &\quad + \sum_{i=1}^k X^i \circ \varpi(\xi) \frac{d}{ds} \langle \eta(s), s_i \circ \varpi(\eta(s)) \rangle \Big|_{s=0}. \end{aligned}$$

Since $X_i \circ \varpi(\xi) = 0$, the last terms vanish, so we obtain

$$\langle d\Phi_X(\xi), W \rangle = \sum_{i=1}^k \langle d(X^i \circ \varpi)(\xi), T_\xi \varpi(W) \rangle \langle \xi, s_i \circ \varpi(\xi) \rangle.$$

Comparing Equation (*) with that last equation, we obtain

$$-\langle df(\varpi(\xi)), T_\xi \varpi(W) \rangle = \sum_{i=1}^k \langle d(X^i \circ \varpi)(\xi), T_\xi \varpi(W) \rangle \langle \xi, s_i \circ \varpi(\xi) \rangle.$$

Since that equality holds for all $W \in T_\xi E^*$, we have

$$df \circ \varpi(\xi) = - \sum_{i=1}^k \langle \xi, s_i \circ \varpi(\xi) \rangle d(X^i \circ \varpi)(\xi). \quad (**)$$

Now, for any smooth section $Y \in A^1(M, E)$, we have on U

$$\{X, Y\} = \sum_{i=1}^k X^i \{s_i, Y\} - \sum_{i=1}^k \langle dX^i, \rho \circ Y \rangle s_i.$$

Since $X^i \circ \varpi(\xi) = 0$, we obtain

$$\Phi_{\{X, Y\}}(\xi) = - \sum_{i=1}^k \langle \xi, s_i \circ \varpi(\xi) \rangle \langle d(X^i \circ \varpi)(\xi), \rho \circ Y(\varpi(\xi)) \rangle.$$

From Equation (**) and that last equation, we finally obtain

$$\Phi_{\{X, Y\}}(\xi) = \langle df, \rho \circ Y \rangle \circ \varpi(\xi)$$

and our proof is complete. \boxtimes

5.3.2 Theorem. *Let (E, π, M, ρ) be a Lie algebroid and (E^*, ϖ, M) its dual bundle. To each smooth section $X \in A^1(M, E)$, we associate a smooth function Φ_X on E^* by setting*

$$\Phi_X(\xi) = \langle \xi, X \circ \varpi(\xi) \rangle.$$

There exists on E^ a unique Poisson structure such that, for each pair (X, Y) of smooth sections of π ,*

$$\{\Phi_X, \Phi_Y\} = \Phi_{\{X, Y\}},$$

the bracket in the left hand side being the Poisson bracket of functions on E^* , and the bracket in the right hand side the bracket of sections of the Lie algebroid (E, π, M, ρ) .

Proof:

If such a Poisson structure exists, it must be such that for any $X \in A^1(M, E)$, f and $g \in C^\infty(M, \mathbb{R})$,

$$\{\Phi_X, g \circ \varpi\} = (\mathcal{L}(\rho \circ X)g) \circ \varpi, \quad \{f \circ \varpi, g \circ \varpi\} = 0.$$

These identities are consequences of the property of the Lie algebroid bracket:

$$\{fX, gY\} = fg\{X, Y\} + (f\mathcal{L}(\rho \circ X)g)Y - (g\mathcal{L}(\rho \circ Y)f)X,$$

which implies

$$\Phi_{\{fX, gY\}} = (fg \circ \varpi)\Phi_{\{X, Y\}} + (f\mathcal{L}(\rho \circ X)g) \circ \varpi \Phi_Y - (g\mathcal{L}(\rho \circ Y)f) \circ \varpi \Phi_X.$$

Now we observe that for any $\xi \in E^*$, η_1 and $\eta_2 \in T_\xi^*E^*$, there exist (non unique) pairs (X_1, f_1) and (X_2, f_2) , with X_1 and $X_2 \in A^1(M, E)$, f_1 and $f_2 \in C^\infty(M, \mathbb{R})$, such that

$$\eta_1 = d(\Phi_{X_1} + f_1 \circ \varpi)(\xi), \quad \eta_2 = d(\Phi_{X_2} + f_2 \circ \varpi)(\xi).$$

The Poisson bivector Λ on E^* is therefore given by the equation

$$\begin{aligned} \Lambda(\xi)(\eta_1, \eta_2) &= \{\Phi_{X_1} + f_1 \circ \varpi, \Phi_{X_2} + f_2 \circ \varpi\}(\xi) \\ &= \Phi_{\{X_1, X_2\}}(\xi) + (\mathcal{L}(\rho \circ X_1)f_2) \circ \varpi(\xi) - (\mathcal{L}(\rho \circ X_2)f_1) \circ \varpi(\xi). \end{aligned}$$

This proves that if such a Poisson structure exists, it is unique. In order to prove its existence we still have to check that the right hand side of the above formula depends only on η_1 and η_2 , not on the particular choices we have made for (X_1, f_1) and (X_2, f_2) . Since that member depends bilinearly on (X_1, f_1) and (X_2, f_2) and is skew-symmetric, it is enough to prove that if $X_1 \in A^1(M, E)$ and $f_1 \in C^\infty(M, \mathbb{R})$ are such that

$$d(\Phi_{X_1} + f_1 \circ \varpi)(\xi) = 0,$$

then for any $X_2 \in A^1(M, E)$ and $f_2 \in C^\infty(M, \mathbb{R})$,

$$\Phi_{\{X_1, X_2\}}(\xi) + (\mathcal{L}(\rho \circ X_1)f_2) \circ \varpi(\xi) - (\mathcal{L}(\rho \circ X_2)f_1) \circ \varpi(\xi) = 0.$$

But that equality follows immediately from Lemma 5.3.1. We have proven the existence of the bivector Λ , and the formula used for its definition proves that it is smooth.

Finally, we observe that the Poisson bracket defined by means of Λ , when restricted to functions on E^* of the type $\Phi_X + f \circ \varpi$, with $X \in A^1(M, E)$ and $f \in C^\infty(M, \mathbb{R})$, satisfies the Jacobi identity. Therefore, for any $\xi \in E^*$, η_1, η_2 and $\eta_3 \in T_\xi^*E^*$ which are the differentials, at ξ , of functions of the type $\Phi_X + f \circ \varpi$, the Schouten bracket $[\Lambda, \Lambda]$ satisfies $[\Lambda, \Lambda](\xi)(\eta_1, \eta_2, \eta_3) = 0$. Since all elements in $T_\xi^*E^*$ are differentials, at ξ , of functions of the type $\Phi_X + f \circ \varpi$, we have proven the $[\Lambda, \Lambda]$ vanishes identically, in other words, that Λ is a Poisson bivector. \boxtimes

5.3.3 Proposition. *Let (E, π, M, ρ) be a Lie algebroid and (E^*, ϖ, M) its dual bundle. The Poisson structure on E^* defined in Theorem 5.3.2 has the following properties:*

1. For any $X \in A^1(M, E)$, f and $g \in C^\infty(M, \mathbb{R})$,

$$\{\Phi_X, g \circ \varpi\} = (\mathcal{L}(\rho \circ X)g) \circ \varpi, \quad \{f \circ \varpi, g \circ \varpi\} = 0,$$

where Φ_X is the smooth function on M associated to the smooth section X as indicated in Theorem 5.3.2.

2. The transpose ${}^t\rho : T^*M \rightarrow E^*$ of the anchor map $\rho : E \rightarrow TM$ is a Poisson map (the cotangent bundle being endowed with the Poisson structure associated to its canonical symplectic structure).

3. The Poisson structure on E^* is homogeneous with respect to the vector field Z which generates the homotheties in the fibres of (E^*, ϖ, M) , which means that its Poisson bivector Λ satisfies

$$[Z, \Lambda] = -\Lambda.$$

Proof:

We have proven Properties 1 in the proof of Theorem 5.3.2. In order to prove Property 2, we must prove that for all pairs (h_1, h_2) of smooth functions on E^* ,

$$\{h_1 \circ {}^t\rho, h_2 \circ {}^t\rho\} = \{h_1, h_2\} \circ {}^t\rho,$$

the bracket in the left hand side being the Poisson bracket of functions on T^*M , and the bracket in the right hand side the Poisson bracket of functions on E^* . It is enough to check that property when h_1 and h_2 are of the type Φ_X , where $X \in A^1(M, E)$, or of the type $f \circ \varpi$, with $f \in C^\infty(M, \mathbb{R})$, since the differentials of functions of these two types generate T^*E^* . For $h_1 = \Phi_X$ and $h_2 = \Phi_Y$, with X and $Y \in A^1(M, E)$, and $\zeta \in T^*M$, we have

$$\begin{aligned} \{\Phi_X, \Phi_Y\} \circ {}^t\rho(\zeta) &= \Phi_{\{X, Y\}} \circ {}^t\rho(\zeta) \\ &= \langle {}^t\rho(\zeta), \{X, Y\} \circ \varpi \circ {}^t\rho(\zeta) \rangle \\ &= \langle \zeta, \rho \circ \{X, Y\} \circ \pi_M(\zeta) \rangle \\ &= \langle \zeta, [\rho \circ X, \rho \circ Y] \circ \pi_M(\zeta) \rangle, \end{aligned}$$

since the canonical projection $\pi_M : T^*M \rightarrow M$ satisfies $\varpi \circ {}^t\rho = \pi_M$. But let us recall a well known property of the Poisson bracket of functions on T^*M ([16], exercise 17.5 page 182). To any vector field \widehat{X} on M , we associate the function $\Psi_{\widehat{X}}$ on T^*M by setting, for each $\zeta \in T^*M$,

$$\Psi_{\widehat{X}}(\zeta) = \langle \zeta, \widehat{X} \circ \pi_M(\zeta) \rangle.$$

Then, for any pair $(\widehat{X}, \widehat{Y})$ of vector fields on M ,

$$\{\Psi_{\widehat{X}}, \Psi_{\widehat{Y}}\} = \Psi_{[\widehat{X}, \widehat{Y}]}$$

By using $\pi_M = \varpi \circ {}^t\rho$, we easily see that for each $X \in A^1(M, E)$,

$$\Psi_{\rho \circ X} = \Phi_X \circ {}^t\rho.$$

Returning to our pair of sections X and $Y \in A^1(M, E)$, we see that

$$\{\Phi_X, \Phi_Y\} \circ {}^t\rho(\zeta) = \Psi_{[\rho \circ X, \rho \circ Y]}(\zeta) = \{\Psi_{\rho \circ X}, \Psi_{\rho \circ Y}\}(\zeta) = \{\Phi_X \circ {}^t\rho, \Phi_Y \circ {}^t\rho\}(\zeta).$$

Now for $h_1 = \Phi_X$ and $h_2 = f \circ \varpi$, with $X \in A^1(M, E)$ and $f \in C^\infty(M, \mathbb{R})$, we have

$$\begin{aligned} \{\Phi_X \circ {}^t\rho, f \circ \varpi \circ {}^t\rho\} &= \{\Psi_{\rho \circ X}, f \circ \pi_M\} \\ &= \mathcal{L}(\rho \circ X) f \circ \pi_M \\ &= \mathcal{L}(\rho \circ X) f \circ \varpi \circ {}^t\rho \\ &= \{\Phi_X, f \circ \varpi\} \circ {}^t\rho. \end{aligned}$$

Similarly, for $h_1 = f \circ \varpi$ and $h_2 = g \circ \varpi$, we have

$$\{f \circ \varpi, g \circ \varpi\} \circ {}^t\rho = 0 = \{f \circ \pi_M, g \circ \pi_M\} = \{f \circ \varpi \circ {}^t\rho, g \circ \varpi \circ {}^t\rho\}.$$

Property 2 is proven. Let us now prove Property 3. The vector field Z on the vector bundle (E^*, ϖ, M) generates the homotheties in the fibres, therefore its reduced flow is $(t, \xi) \mapsto H_t(\xi) = e^t \xi$, with $t \in \mathbb{R}$, $\xi \in E^*$. For any smooth section $X \in A^1(M, E)$ and any $t \in \mathbb{R}$, we have

$$(H_t^* \Phi_X)(\xi) = \Phi_X \circ H_t(\xi) = e^t \Phi_X(\xi).$$

Therefore, for X and $Y \in A^1(M, E)$,

$$H_t^*(\Lambda(d\Phi_X, d\Phi_Y)) = \{\Phi_X, \Phi_Y\} \circ H_t = \Phi_{\{X, Y\}} \circ H_t = e^t \Phi_{\{X, Y\}}.$$

But we may also write

$$H_t^*(\Lambda(d\Phi_X, d\Phi_Y)) = (H_t^* \Lambda)(H_t^* d\Phi_X, H_t^* d\Phi_Y) = e^{2t} (H_t^* \Lambda)(d\Phi_X, d\Phi_Y)$$

Since the differentials of functions of the type Φ_X generate T^*E , except along its zero section, this result proves that, except maybe along the zero section, we have

$$H_t^*(\Lambda) = e^{-t} \Lambda.$$

In fact that equality holds everywhere, since Λ vanishes on the zero section of $(E^* \varpi, M)$. Finally,

$$[Z, \Lambda] = \mathcal{L}(Z)\Lambda = \frac{d}{dt} H_t^*(\Lambda) \Big|_{t=0} = -\Lambda,$$

and our proof is complete. \square

References

- [1] C. Albert, P. Dazord, *Théorie des groupoïdes symplectiques, Ch. II*, Publ. Dépt. Math., Univ. Lyon I, nouvelle série (1990), 27–99.
- [2] M. Bangoura and Y. Kosmann-Schwarzbach, *Équation de Yang-Baxter dynamique classique et algébroides de Lie*, C.R. Acad. Sci. Paris 327, I, 1998, 541–546.
- [3] A. Cannas da Silva and A. Weinstein, *Geometric models for noncommutative algebras*, Berkeley Mathematics Lecture Notes 10, Amer. Math. Soc., Providence, 1999.
- [4] A. Coste, P. Dazord and A. Weinstein, *Groupoïdes symplectiques*, Publ. Dépt. Math., Univ. Lyon I, 2/A (1987), 1–64.
- [5] P. Dazord and D. Sondaz, *Variétés de Poisson, Algébroides de Lie*, Publ. Dépt. Math., Univ. Lyon I, nouvelle série 1/B (1988), 1–68.
- [6] W. Greub, S. Halperin and R. Vanstone, *Connections, curvature and cohomology*, vol I, Academic Press, New York, London, 1972.
- [7] J. Huebschmann, *Poisson cohomology and quantization*, J. Reine Angew. Math. 408 (1990), 57–113.
- [8] D. Iglesias and J.C. Marrero, *Generalized Lie bialgebroids and Jacobi structures*, preprint, Universidad de la Laguna, Spain, 2000.
- [9] M. Karasev, *Analogues of the objects of Lie group theory for nonlinear Poisson brackets*, Math. USSR Izvest. 28 (1987), 497–527.
- [10] A. Kirillov, *Local Lie algebras*, Russian Math. Surveys 31 (1976), 55–75.
- [11] Y. Kosmann-Schwarzbach, *Exact Gerstenhaber algebras and Lie bialgebroids*, Acta Appl. Math. 41 (1995), 153–165.

-
- [12] J.-L. Koszul, *Homologie et cohomologie des algèbres de Lie*, Bull. Soc. Math. France 78 (1950), 65–127.
- [13] J.-L. Koszul, *Crochet de Schouten-Nijenhuis et cohomologie*, Astérisque, hors série, 1985, 257–271.
- [14] P. Libermann, *On symplectic and contact groupoids*, Differential Geometry and its applications, Proc. Conf. Opava, August 24–28, 1992, Silesian University, Opava, 1993, 29–45.
- [15] P. Libermann, *Lie algebroids and mechanics*, Archivum mathematicum 32 (1996), 1147–162.
- [16] P. Libermann and Ch.-M. Marle, *Symplectic Geometry and Analytical Mechanics*, Kluwer, Dordrecht, 1987.
- [17] A. Lichnerowicz, *Les variétés de Poisson et leurs algèbres de Lie associées*, J. Differential Geometry 12 (1977), 253–300.
- [18] A. Lichnerowicz, *Les variétés de Jacobi et leurs algèbres de Lie associées*, J. Math. pures et appl. 57 (1978), 453–488.
- [19] K.C.H. Mackenzie, *Lie groupoids and Lie algebroids in differential geometry*, London Math. Soc. Lecture notes series 124, Cambridge University Press, Cambridge (1987).
- [20] K.C.H. Mackenzie and P. Xu, *Lie bialgebroids and Poisson groupoids*, Duke Math. J. 73 (1994), 415–452.
- [21] F. Magri and C. Morosi, *A geometrical characterization of integrable Hamiltonian systems through the theory of Poisson-Nijenhuis manifolds*, Quaderno S. 19 (1984), Università di Milano.
- [22] A. Nijenhuis, *Jacobi-type identities for bilinear differential concomitants of certain tensor fields*, Indag. Math. 17 (1955), 390–403.

-
- [23] J. Pradines, Théorie de Lie pour les groupoïdes différentiables. Calcul différentiel dans la catégorie des groupoïdes infinitésimaux, C.R. Acad. Sci. Paris 264 A (1967), 245-248.
- [24] J. A. Schouten, *On the differential operators of first order in tensor calculus*, Convegno Intern. Geom. Diff. Italia, Cremonese, Roma (1953), 1-7.
- [25] I. Vaisman, *Lectures on the Geometry of Poisson Manifolds*, Birkhäuser, Basel, Boston, Berlin (1994).
- [26] A. Weinstein, *The local structure of Poisson manifolds*, J. Differential Geometry 18 (1983), 523-557.
- [27] A. Weinstein, *Symplectic groupoids and Poisson manifolds*, Bull. Amer. Math. Soc. 16 (1987), 101-103.
- [28] A. Weinstein, *Groupoids: unifying internal and external symmetry, a tour through some examples*, Notices of the Amer. Math. Soc. 43 (1996), 744-752.
- [29] P. Xu, *Poisson cohomology of regular Poisson manifolds*, Ann. Inst. Fourier, Grenoble, 42, 4 (1992), 967-988.