Reduction of bihamiltonian manifolds and recursion operators

Joana Margarida Nunes da Costa *
and Charles-Michel Marle**

Acknowledgements. The authors, particularly C.-M. M., are indebted to Professor Franco Magri for many helpful discussions.

All the manifolds, maps, differential forms, vector and tensor fields, considered in what follows are assumed to be differentiable of class C^∞.

1. Poisson manifolds and the Schouten-Nijenhuis bracket

1.1. Some general properties and notations. Let Λ be a bivector, that means a 2-times contravariant skew-symmetric tensor field, on a smooth manifold M. It defines a vector bundle map $\Lambda^\sharp : T^*M \to TM$, such that, for any $x \in M$, α and $\beta \in T^*_x M$,

$$\langle \beta, \Lambda^\sharp(\alpha)\rangle = \Lambda_x(\alpha, \beta)\,.$$

We can therefore associate with any differential 1-form α, a vector field $\Lambda^\sharp(\alpha)$.

For any pair of smooth functions f and $g \in C^\infty(M, \mathbf{R})$, we set

$$\{f, g\} = \Lambda(df, dg) = i\bigl(\Lambda^\sharp(df)\bigr)dg = -i\bigl(\Lambda^\sharp(dg)\bigr)df\,.$$

We shall say that $(f, g) \mapsto \{f, g\}$ is the *bracket* associated with Λ. It satisfies, for all f, g, g_1 and $g_2 \in C^\infty(M, \mathbf{R})$,

$$\{f, g\} = -\{g, f\}\,, \quad \{f, g_1 g_2\} = \{f, g_1\}g_2 + g_1\{f, g_2\}\,.$$

* Universidade de Coimbra, Departamento de Matemática, Apartado 3008, 3000 Coimbra, Portugal. Email: jmcosta@mat.uc.pt
** Université Pierre et Marie Curie, Institut de Mathématiques, 4, place Jussieu, 75252 Paris cedex 05, France. Email: marle@mathp6.jussieu.fr

1.2. Definition. A bivector Λ on a smooth manifold M is said to be a *Poisson tensor* if the bracket associated with it satisfies the Jacobi identity, that is, if for all f, g and $h \in C^\infty(M, \mathbf{R})$,

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0\,.$$

In that case, the manifold M equipped with Λ is called a *Poisson manifold* and denoted by (M, Λ); the bracket associated with Λ is called a *Poisson bracket* and, for any smooth function f on M, the vector field $\Lambda^\sharp(df)$ is called the *Hamiltonian vector field* associated with f.

Let us recall the well known result (Lichnerowicz [8]):

1.3. Proposition. *Let Λ be a bivector on a smooth manifold M. The following three properties are equivalent:*

(i) *the bivector Λ is a Poisson tensor;*

(ii) *the Schouten-Nijenhuis bracket $[\Lambda, \Lambda]$ of Λ with itself vanishes;*

(iii) *for all f and $g \in C^\infty(M, \mathbf{R})$,*

$$\big[\Lambda^\sharp(df), \Lambda^\sharp(dg)\big] = \Lambda^\sharp\big(d\{f, g\}\big)\,.$$

When these equivalent properties are satisfied, $C^\infty(M, \mathbf{R})$, with the Poisson bracket as composition law, is a Lie algebra and the map $f \mapsto \Lambda^\sharp(df)$ is a Lie algebra homomorphism.

1.4. Some properties of the Schouten-Nijenhuis bracket.

Let us recall the definition and some properties of the Schouten-Nijenhuis bracket [15, 13, 14, 5]. We use in the following the conventions of Koszul [5], which may differ by signs and even numerical factors from those used in [8] or [9].

Let M be a smooth manifold. We denote by $A(M) = \oplus_p A^p(M)$ and by $\Omega(M) = \oplus_p \Omega^p(M)$ the graded algebras, repectively, of skew-symmetric contravariant tensor fields, and of differential forms on M.

The Schouten-Nijenhuis bracket is a bilinear composition law on $A(M)$, denoted by $(P, Q) \mapsto [P, Q]$, which satisfies the following properties:

(i) if $P \in A^p(M)$ and $Q \in A^q(M)$, then $[P, Q] \in A^{p+q-1}(M)$;

(ii) if $X \in A^1(M)$ is a vector field on M, then for any $Q \in A(M)$, $[X, Q]$ is the Lie derivative $\mathcal{L}(X)Q$ of Q with respect to X;

(iii) if $P \in A^p(M)$ and $Q \in A^q(M)$, then

$$[P, Q] = -(-1)^{(p-1)(q-1)}[Q, P]\,;$$

(iv) for any $P \in A^p(M)$, the map $Q \mapsto [P, Q]$ is a derivation of degree $p - 1$ of the graded exterior algebra $A(M)$ (with the wedge product as composition law); in other words, for any $Q_1 \in A^{q_1}(M)$ and $Q_2 \in A(M)$,

$$[P, Q_1 \wedge Q_2] = [P, Q_1] \wedge Q_2 + (-1)^{(p-1)q_1} Q_1 \wedge [P, Q_2]\,.$$

These four properties determine the Schouten-Nijenhuis bracket in a unique way.

The Schouten-Nijenhuis bracket has several other remarkable properties. In particular,

(v) it satisfies the graded Jacobi identity

$$(-1)^{(p-1)(r-1)}\big[P,[Q,R]\big]+(-1)^{(q-1)(p-1)}\big[Q,[R,P]\big] \\ +(-1)^{(r-1)(q-1)}\big[R,[P,Q]\big]=0\,,$$

where $P\in A^p(M)$, $Q\in A^q(M)$ and $R\in A^r(M)$; therefore, $A(M)$, with the Schouten-Nijenhuis bracket as composition law, is a graded Lie algebra.

There is another important property of the Schouten-Nijenhuis bracket, which links it with the exterior differentiation operator d. We must first recall some conventions about the exterior algebras $A(M)$ and $\Omega(M)$. There exists a bilinear map from $\Omega(M)\times A(M)$ into $C^\infty(M,\mathbf{R})$, called the *coupling map*, denoted by $(\eta,P)\mapsto\langle\eta,P\rangle$. We define it first when $\eta=\eta_1\wedge\cdots\wedge\eta_q\in\Omega^q(M)$ is a decomposable q-form (the $\eta_i\in\Omega^1(M)$ being 1-forms on M), and when $P=P_1\wedge\cdots\wedge P_p\in A^p(M)$ is a decomposable p-vector (the $P_i\in A^1(M)$ being vector fields on M). Then

$$\big\langle\eta_1\wedge\cdots\wedge\eta_q,P_1\wedge\cdots\wedge P_p\big\rangle=\begin{cases}0 & \text{if } p\neq q,\\ \det\big(\langle\eta_i,P_j\rangle\big) & \text{if } p=q.\end{cases}$$

We have denoted by $\langle\eta_i,P_j\rangle=i(P_j)\eta_i$ the usual coupling of a 1-form η_i with a vector field P_j.

The coupling map, which is local, can then be extended by bilinearity to the whole of $\Omega(M)\times A(M)$, since locally any differential q-form and any p-vector are sums of decomposable elements.

We define next a bilinear map from $A(M)\times\Omega(M)$ into $\Omega(M)$, called the *inner product* of a form by a multivector, and denoted by $(P,\eta)\mapsto i(P)\eta$. When $P\in A^p(M)$ and $\eta\in\Omega^q(M)$ are homogeneous of degrees p and q, respectively, then $i(P)\eta\in\Omega^{q-p}$ is the unique $(q-p)$-form such that, for any $R\in A^{q-p}(M)$,

$$\big\langle i(P)\eta,R\big\rangle=(-1)^{(p-1)p/2}\langle\eta,P\wedge R\rangle\,.$$

We observe that when $p>q$, $i(P)\eta$ vanishes.

The definition of the inner product $i(P)\eta$ can be readily extended by bilinearity when P and η are not homogeneous.

When $P\in A^p(M)$ is a fixed homogeneous p-tensor, then $i(P):\eta\mapsto i(P)\eta$ is a graded endomorphism of the graded exterior algebra $\Omega(M)$, of degree $-p$.

The numerical coefficient $(-1)^{(p-1)p/2}$ in the above formula may seem unnatural; however it is such that when $X\in A^1(M)$ is a vector field, then $i(X)$ is the usual inner product by the vector field X; it is also such that, for any $P\in A^p(M)$ and $Q\in A^q(M)$,

$$i(P)i(Q)=i(P\wedge Q)\,.$$

Let D_1 and D_2 be two graded endomorphisms of the exterior algebra $\Omega(M)$, of degrees d_1 and d_2, respectively. We define their *bracket*, denoted by $[D_1, D_2]$, as the graded endomorphism of $\Omega(M)$, of degree $d_1 + d_2$,

$$[D_1, D_2] = D_1 D_2 - (-1)^{d_1 d_2} D_2 D_1 .$$

We can now indicate the next property of the Schouten-Nijenhuis bracket:

(vi) let $P \in A^p(M)$, $Q \in A^q(M)$, $[P, Q]$ their Schouten-Nijenhuis bracket; let $i(P)$, $i(Q)$ and $i\bigl([P, Q]\bigr)$ be the inner products by P, Q and $[P, Q]$, respectively, considered as graded endomorphisms of $\Omega(M)$; then we have

$$i\bigl([P, Q]\bigr) = \bigl[[i(P), d], i(Q)\bigr] ,$$

where d denotes the exterior differentiation operator.

The above property determines the Schouten-Nijenhuis bracket in a unique way, and could be used as a definition.

1.5. Applications of the Schouten-Njenhuis bracket to Poisson manifolds.

Let Λ be a bivector on a smooth manifold M. Using the definitions of the bracket associated with Λ, of the coupling map and of the inner product given above we have, for all f and $g \in C^\infty(M, \mathbf{R})$,

$$\{f, g\} = i\bigl(\Lambda^\sharp(df)\bigr)dg = \Lambda(df, dg) = \bigl\langle df \wedge dg, \Lambda\bigr\rangle = -i(\Lambda)(df \wedge dg) .$$

By property 1.4.(vi), we have

$$i\bigl([\Lambda, f]\bigr)dg = \bigl[[i(\Lambda), d], i(f)\bigr]dg = i(\Lambda)\bigl(df \wedge dg\bigr) .$$

Therefore,

$$\Lambda^\sharp(df) = -[\Lambda, f] = -[f, \Lambda] , \qquad \{f, g\} = -\bigl[[\Lambda, f], g\bigr] = \bigl[[\Lambda, g], f\bigr] .$$

Similarly, for all f, g and $h \in C^\infty(M, \mathbf{R})$,

$$\begin{aligned}
[\Lambda, \Lambda](df \wedge dg \wedge dh) &= -i\bigl([\Lambda, \Lambda]\bigr)(df \wedge dg \wedge dh) = -\bigl[[i(\Lambda), d], i(\Lambda)\bigr](df \wedge dg \wedge dh)\\
&= -2i(\Lambda)di(\Lambda)\bigl(df \wedge dg \wedge dh\bigr)\\
&= 2i(\Lambda)\bigl(d\{f, g\} \wedge dh + d\{g, h\} \wedge df + d\{h, f\} \wedge dg\bigr)\\
&= -2\bigl(\bigl\{\{f, g\}, h\bigr\} + \bigl\{\{g, h\}, f\bigr\} + \bigl\{\{h, f\}, g\bigr\}\bigr)\\
&= 2\bigl(\bigl\{f, \{g, h\}\bigr\} + \bigl\{g, \{h, f\}\bigr\} + \bigl\{h, \{f, g\}\bigr\}\bigr)\\
&= 2\bigl(\bigl[\Lambda^\sharp(df), \Lambda^\sharp(dg)\bigr] - \Lambda^\sharp\bigl(d\{f, g\}\bigr)\bigr).h .
\end{aligned}$$

Proposition 1.3 follows easily from this formula.

Let Λ_0 and Λ_1 be two bivectors on the smooth manifold M. For all f and $g \in C^\infty(M, \mathbf{R})$, we denote by $\{f, g\}_0 = \Lambda_0(df, dg)$ and $\{f, g\}_1 = \Lambda_1(df, dg)$ the brackets defined by Λ_0 and Λ_1, respectively. We set

$$X_f = \Lambda_0(df)\,, \qquad Y_f = \Lambda_1(df)\,.$$

Then by a calculation similar to that above (or by polarizing the above formulas) we have, for all f, g and $h \in C^\infty(M, \mathbf{R})$,

$$\begin{aligned}[\Lambda_0, \Lambda_1](df, dg, dh) &= \big\{f, \{g, h\}_0\big\}_1 + \big\{g, \{h, f\}_0\big\}_1 + \big\{h, \{f, g\}_0\big\}_1 \\ &\quad + \big\{f, \{g, h\}_1\big\}_0 + \big\{g, \{h, f\}_1\big\}_0 + \big\{h, \{f, g\}_1\big\}_0 \\ &= \big([X_f, Y_g] + [Y_f, X_g] - X_{\{f,g\}_1} - Y_{\{f,g\}_0}\big).h\,.\end{aligned}$$

We can also write

$$\begin{aligned}[\Lambda_0, \Lambda_1](df, dg, dh) &= \big\{f, \{g, h\}_0\big\}_1 + \big\{g, \{h, f\}_0\big\}_1 + \big\{h, \{f, g\}_0\big\}_1 \\ &\quad + \big\{f, \{g, h\}_1\big\}_0 + \big\{g, \{h, f\}_1\big\}_0 + \big\{h, \{f, g\}_1\big\}_0 \\ &= Y_f.\{g, h\}_0 - \{g, Y_f.h\}_0 - \{Y_f.g, h\}_0 \\ &\quad + X_f.\{g, h\}_1 - \{g, X_f.h\}_1 - \{X_f.g, h\}_1 \\ &= \big(\mathcal{L}(Y_f)\Lambda_0 + \mathcal{L}(X_f)\Lambda_1\big)(dg, dh)\,.\end{aligned}$$

These formulas will be used in the next section.

2. Bihamiltonian manifolds

2.1. Definition. Two linearly independent Poisson tensors Λ_0 and Λ_1 defined on the same smooth manifold M are said to be *compatible* when $\Lambda_0 + \Lambda_1$ is again a Poisson tensor. In such a case the manifold M, equipped with the two Poisson tensors Λ_0 and Λ_1, is called a *bihamiltonian manifold*, and denoted by $(M, \Lambda_0, \Lambda_1)$.

2.2. Proposition. *Let M be a smooth manifold equipped with two linearly independent Poisson tensors Λ_0 and Λ_1. With the notations indicated at the end of the preceding section, the following properties are equivalent:*

(i) *the Poisson tensors Λ_0 and Λ_1 are compatible;*
(ii) *the Schouten-Nijenhuis bracket $[\Lambda_0, \Lambda_1]$ vanishes;*
(iii) *for all f, g and $h \in C^\infty(M, \mathbf{R})$,*

$$\begin{aligned}&\big\{f, \{g, h\}_0\big\}_1 + \big\{g, \{h, f\}_0\big\}_1 + \big\{h, \{f, g\}_0\big\}_1 \\ &\quad + \big\{f, \{g, h\}_1\big\}_0 + \big\{g, \{h, f\}_1\big\}_0 + \big\{h, \{f, g\}_1\big\}_0 = 0\,;\end{aligned}$$

(iv) *for all* f *and* $g \in C^\infty(M, \mathbf{R})$,

$$[X_f, Y_g] + [Y_f, X_g] - X_{\{f,g\}_1} - Y_{\{f,g\}_0} = 0\,;$$

(v) *for all* $f \in C^\infty(M, \mathbf{R})$,

$$\mathcal{L}(Y_f)\Lambda_0 + \mathcal{L}(X_f)\Lambda_1 = 0\,.$$

When these properties are satisfied, any two linearly independent linear combinations of Λ_0 *and* Λ_1 *are compatible Poisson structures.*

Proof. Taking into account the bilinearity of the Schouten-Nijenhuis bracket, it follows immediately from Proposition 1.3 and the formulas in 1.5. □

The importance of bihamiltonian manifolds arises from the fact that many completely integrable systems are vector fields on a bihamiltonian manifold, which are Hamiltonian with respect to both Poisson structures. Euler's equations, which govern the motion of a rigid body with a fixed point, are of that type.

2.3. Example: Euler's equations for the rigid body. Let us assume for simplicity that the three principal moments of inertia I_1, I_2 and I_3 of the body, at its fixed point, are strictly positive and distinct. Euler's equation are

$$\begin{aligned}\frac{dx_1}{dt} &= (I_2^{-1} - I_3^{-1})x_2x_3\,,\\ \frac{dx_2}{dt} &= (I_3^{-1} - I_1^{-1})x_3x_1\,,\\ \frac{dx_3}{dt} &= (I_1^{-1} - I_2^{-1})x_1x_2\,.\end{aligned}$$

We have denoted by x_1, x_2, x_3 the components of the angular momentum of the body at its fixed point, in the inertial orthogonal frame, attached to the moving body, made by the eigenvectors of the inertia operator. We set:

$$\begin{aligned}\Lambda_0 &= -\left(x_1\frac{\partial}{\partial x_2}\wedge\frac{\partial}{\partial x_3} + x_2\frac{\partial}{\partial x_3}\wedge\frac{\partial}{\partial x_1} + x_3\frac{\partial}{\partial x_1}\wedge\frac{\partial}{\partial x_2}\right)\,,\\ \Lambda_1 &= I_1^{-1}x_1\frac{\partial}{\partial x_2}\wedge\frac{\partial}{\partial x_3} + I_2^{-1}x_2\frac{\partial}{\partial x_3}\wedge\frac{\partial}{\partial x_1} + I_3^{-1}x_3\frac{\partial}{\partial x_1}\wedge\frac{\partial}{\partial x_2}\,,\\ H &= \frac{1}{2}(I_1^{-1}x_1^2 + I_2^{-1}x_2^2 + I_3^{-1}x_3^2)\,,\\ L &= \frac{1}{2}(x_1^2 + x_2^2 + x_3^2)\,.\end{aligned}$$

We observe that

$$\Lambda_0^\sharp(dH) = \Lambda_1^\sharp(dL)\,,$$

and that the differential equations defined by that vector field, which is Hamiltonian with respect to both Λ_0 and Λ_1, are Euler's equations. In [4], Holm and Marsden have used the fact that Euler's equations are Hamiltonian with respect to any nonvanishing linear combination of Λ_0 and Λ_1 to derive remarkable properties of that completely integrable system.

3. Poisson-Nijenhuis manifolds

3.1. The Nijenhuis torsion. Let N be a tensor field of type $(1,1)$ (one time contravariant and one time covariant) on a smooth manifold M. We shall consider $N : TM \to TM$ as a vector bundle map of the tangent bundle TM into itself, and we shall denote by ${}^tN : T^*M \to T^*M$ the transpose map. Nijenhuis [12] has shown that by setting, for any pair (X,Y) of vector fields on M,

$$\mathcal{T}(N)(X,Y) = [NX,NY] - N\big([NX,Y] + [X,NY] - N[X,Y]\big)\,,$$

we define a vector bundle map $\mathcal{T}(N) : TM \oplus TM \to TM$ or, in other words, a tensor field $\mathcal{T}(N)$ of type $(1,2)$ (one time contravariant and two-times covariant), called the *Nijenhuis torsion* of N.

3.2. The recursion operator. Let Λ_0 be a Poisson tensor and N a tensor field of type $(1,1)$ on a smooth manifold M. With the same conventions as above, we assume that

$$N \circ \Lambda_0^\sharp = \Lambda_0^\sharp \circ {}^tN\,,$$

and we set

$$\Lambda_1^\sharp = N \circ \Lambda_0^\sharp = \Lambda_0^\sharp \circ {}^tN\,.$$

The bundle map $\Lambda_1^\sharp : T^*M \to TM$ is associated with a 2-times contravariant skew-symmetric tensor field Λ_1. For any smooth functions f and g on M, we set

$$X_f = \Lambda_0^\sharp(df)\,, \quad Y_f = \Lambda_1^\sharp(df) = NX_f\,,$$

and similarly

$$\{f,g\}_0 = \Lambda_0(df,dg) = X_f.g\,, \quad \{f,g\}_1 = \Lambda_1(df,dg) = Y_f.g\,.$$

As seen in 1.5, Λ_1 is a Poisson tensor if and only if, for any pair (f,g) of smooth functions on M,

$$[NX_f, NX_g] = NX_{\{f,g\}_1}\,.$$

Similarly, the Schouten-Nijenhuis bracket $[\Lambda_1, \Lambda_0]$ vanishes if and only if, for any pair (f,g) of smooth functions on M,

$$[NX_f, X_g] + [X_f, NX_g] - X_{\{f,g\}_1} - NX_{\{f,g\}_0} = 0\,.$$

Since Λ_0 is a Poisson tensor, we have

$$X_{\{f,g\}_0} = [X_f, X_g]\,,$$

therefore $[\Lambda_1, \Lambda_0]$ vanishes if and only if, for any pair (f,g) of smooth functions on M,

$$X_{\{f,g\}_1} = [NX_f, X_g] + [X_f, NX_g] - N[X_f, X_g]\,.$$

Therefore, when Λ_1 is a Poisson tensor compatible with Λ_0 we have, for any pair (f,g) of smooth functions on M,

$$[NX_f, NX_g] - N\big([NX_f, X_g] + [X_f, NX_g] - N[X_f, X_g]\big) = 0\,.$$

Using the expression of the Nijenhuis torsion of N, we may state:

3.3. Proposition. *Let Λ_0 be a Poisson tensor and N a tensor field of type $(1,1)$ on a smooth manifold M, such that*

$$N\Lambda_0^\sharp = \Lambda_0^\sharp\, {}^tN\,.$$

We assume that the tensor field Λ_1 associated with the vector bundle map $\Lambda_1^\sharp = N\Lambda_0^\sharp$ satisfies

$$[\Lambda_1, \Lambda_0] = 0\,.$$

Then Λ_1 is a Poisson tensor if and only if the Nijenhuis torsion $\mathcal{T}(N)$ vanishes on the image $\Lambda_0^\sharp(T^*M)$ of $\Lambda_0^\sharp$. In such a case, $(M, \Lambda_0, \Lambda_1 = N\Lambda_0)$ is a bihamiltonian manifold, called a Poisson-Nijenhuis manifold, and the tensor field N is called its recursion operator.

3.4. The Magri-Morosi concomitant. In what follows, M is a smooth manifold and, for any p-times contravariant skew-symmetric tensor field $P \in A^p(M)$, and any family $(\alpha_1, \dots, \alpha_{p-1})$ of $p-1$ Pfaff forms on M, we will denote by $P(\alpha_1, \dots, \alpha_{p-1})$ the unique vector field on M such that, for any Pfaff form α_p on M,

$$\big\langle \alpha_p, P(\alpha_1, \dots, \alpha_{p-1})\big\rangle = P(\alpha_1, \dots, \alpha_{p-1}, \alpha_p)\,.$$

In particular, for $p = 2$, we will write

$$P(\alpha) = P^\sharp(\alpha)\,.$$

Let $P \in A^2(M)$ be a skew-symmetric, two-times contravariant tensor field, and N a tensor field of type $(1,1)$ on the smooth manifold M. We assume that

$$NP^\sharp = P^\sharp\, {}^tN\,.$$

Magri and Morosi [9] have shown that with P and N one can construct a new tensor field $\mathcal{R}(P,N)$ of type $(2,1)$, by setting, for any pair of a 1-form α and a vector field X,

$$\mathcal{R}(P,N)(\alpha,X) = \big(\mathcal{L}(P^\sharp\alpha)N\big)X - P^\sharp\big(\mathcal{L}(X)({}^tN\alpha)\big) + P^\sharp\big(\mathcal{L}(NX)\alpha\big)\,.$$

We will call $\mathcal{R}(P,N)$ the *Magri-Morosi concomitant* of P and N. It satisfies the following properties [9], which link it with the Nijenhuis torsion and the Schouten bracket:

(i) Let NP be the skew-symmetric two-times contravariant tensor field associated with the vector bundle map $NP^\sharp$. It satisfies $N(NP)^\sharp = (NP)^\sharp\,{}^tN$, and for any 1-form α and any vector field X,

$$\mathcal{R}(NP,N)(\alpha,X) = N\mathcal{R}(P,N)(\alpha,X) + \mathcal{T}(N)(X,P^\sharp\alpha)\,.$$

(ii) For any pair (α,β) of 1-forms on M,

$$[NP,NP](\alpha,\beta) = N\big([P,P]({}^tN\alpha,\beta) - 2\mathcal{R}(P,N)(\alpha,P\beta)\big) + 2\mathcal{T}(N)(P\alpha,P\beta)\,.$$

(iii) Let P and $Q \in A^2(M)$ be two 2-times contravariant skew-symmetric tensor fields and N a tensor field of type $(1,1)$ such that

$$NP^\sharp = P^\sharp\,{}^tN\,,\quad NQ^\sharp = Q^{\sharp t}N\,.$$

Then, for any pair (α,β) of Pfaff forms on M,

$$\begin{aligned}[NP,Q](\alpha,\beta) = N[P,Q](\alpha,\beta) - Q\mathcal{R}^*(P,N)(\alpha,\beta)\\ - \mathcal{R}(Q,N)(\beta,P\alpha) + \mathcal{R}(Q,N)(\alpha,P\beta)\,,\end{aligned}$$

where the transpose $\mathcal{R}^*(P,N)$ of $\mathcal{R}(P,N)$ is defined by the formula, in which α and β are any Pfaff forms and X any vector field,

$$\big\langle\mathcal{R}^*(P,N)(\alpha,\beta),X\big\rangle = \big\langle\beta,\mathcal{R}(P,N)(\alpha,X)\big\rangle\,.$$

3.5. Remark. Let us indicate another useful formula involving the Schouten-Nijenhuis bracket and the Nijenhuis torsion, but not the Magri-Morosi concomitant. We shall use the same conventions as in Section 3.4. Let $P \in A^2(M)$ be a two-times contravariant skew-symmetric tensor field and N a tensor field of type $(1,1)$ such that

$$NP^\sharp = P^\sharp\,{}^tN\,.$$

Then

$$[NP,NP] - 2N[NP,P] - N^2[P,P] = 2\mathcal{T}(N)\circ(P,P)\,,$$

where $\mathcal{T}(N) \circ (P, P)$ is the vector bundle map such that, for any pair (α, β) of Pfaff forms,

$$\mathcal{T}(N) \circ (P, P)(\alpha, \beta) = \mathcal{T}(N)(P\alpha, P\beta)\,.$$

This formula can be easily proven by using results indicated in 1.5.

By using the above formulas, we can easily prove the following result, which is a slight generalization of a theorem due to Magri and Morosi [9]:

3.6. Theorem. *Let $(M, \Lambda_0, \Lambda_1 = N\Lambda_0)$ be a Poisson-Nijenhuis manifold. For any integer $k \geq 1$, we set*

$$\Lambda_k^\sharp = N^k \circ \Lambda_0^\sharp\,,$$

and we denote by Λ_k the two-times contravariant skew-symmetric tensor field associated with the vector bundle map $\Lambda_k^\sharp$. Then, for all $k \in \mathbf{N}$, the Λ_k are Poisson tensors and are pairwise compatible. In other words, for all k and $l \in \mathbf{N}$,

$$[\Lambda_k, \Lambda_l] = 0\,.$$

4. Reduction of bihamiltonian manifolds

The reduction theorem of Marsden and Ratiu for Poisson manifolds [11] can be easily extended to bihamiltonian manifolds, as was shown independently by F. Magri (personal communication) and one of the authors (J. M. N. da C.):

4.1. Theorem. *Let $(M, \Lambda_0, \Lambda_1)$ be a bihamiltonian manifold, S a submanifold of M, and D a vector sub-bundle of $T_S M$, which satisfy the following conditions:*

(i) *For any pair (F, G) of smooth functions on M whose differentials dF and dG, restricted to S, vanish on the sub-bundle D, the differentials $d\{F, G\}_0$ and $d\{F, G\}_1$ of the Poisson brackets of F and G, for each Poisson structure Λ_0 and Λ_1, restricted to S, vanish on D.*

(ii) *Let $D^0 \subset T_S^* M$ be the annihilator of D. Then*

$$\Lambda_0^\sharp(D^0) \subset TS + D \quad \text{and} \quad \Lambda_1^\sharp(D^0) \subset TS + D\,.$$

(iii) *The distribution $TS \cap D$ on S is completely integrable; the set $\dot{S}$ of leaves of the foliation defined on S by that distribution is a smooth manifold, and the canonical projection $\pi : S \to \dot{S}$ is a submersion.*

Then there exists on $\dot{S}$ a unique pair $(\dot{\Lambda}_0, \dot{\Lambda}_1)$ of compatible Poisson structures, such that, for any pair (f, g) of smooth functions on $\dot{S}$ and any smooth extensions *(to a neighbourhood of S in M) F of $f\circ\pi$ and G of $g\circ\pi$ whose differentials dF and dG, restricted to S, vanish on D,*

$$\{F,G\}_0\mid_S=\{f,g\}_0\circ\pi\,,\qquad \{F,G\}_1\mid_S=\{f,g\}_1\circ\pi\,.$$

We will say that $(\dot S,\dot\Lambda_0,\dot\Lambda_1)$ is the reduced bihamiltonian manifold defined by S and D.

4.2. Remarks.

1. When $(M,\Lambda_0,\Lambda_1=N\Lambda_0)$ is a Poisson-Nijenhuis manifold, condition (ii) of Theorem 4.1 may be replaced by the slightly more restrictive condition

$$\Lambda_0^\sharp(D^0)\subset TS+D\quad\text{and}\quad N(TS+D)\subset TS+D\,.$$

But the reduced bihamiltonian manifold $(\dot S,\dot\Lambda_0,\dot\Lambda_1)$ is not in general a Poisson-Nijenhuis manifold. However, in many cases the higher order Poisson tensors $\Lambda_k=N^k\Lambda_0$ on M ($k\geq 1$) induce on $\dot S$ higher order Poisson tensors $\dot\Lambda_k$, which are pairwise compatible, although they are not deduced from $\dot\Lambda_0$ by repeated application of a recursion operator. We shall see an example of that construction in the next section.

2. Magri and Morosi [9, 10] considered the following situation: (M,Λ_0,Λ_1) is a bihamiltonian manifold, and S a symplectic leaf (see for example Weinstein [17], Vaisman [16] or [7]) of the Poisson manifold (M,Λ_1). The sub-bundle D of T_SM chosen here is

$$D=\Lambda_0^\sharp(\ker\Lambda_1^\sharp)\mid_S\,.$$

Assuming that $D\cap TS$ is of constant rank, one can easily prove that $D\cap TS$ is completely integrable, and that conditions (i) and (ii) of Theorem 4.1 are satisfied. If in addition the set of leaves of the foliation defined on S by $D\cap TS$ is a smooth manifold, and if the canonical projection $\pi:S\to\dot S$ is a submersion, Theorem 4.1 can be applied.

4.3. A simple case. In view of its application to the Toda lattice, we consider the following simple case. Let (M,Λ_0) be an $2n$-dimensional Poisson manifold whose Poisson tensor Λ_0 is everywhere of rank $2n$; in other words, M is a symplectic manifold, equipped with the Poisson structure canonically associated with its symplectic structure. Let Λ_1 be another Poisson tensor on M, which may not be compatible with Λ_0. Since $\Lambda_0^\sharp:T^*M\to TM$ is an isomorphism, we may set

$$N=\Lambda_1^\sharp\circ(\Lambda_0^\sharp)^{-1}\,.$$

Then N is a tensor field of type $(1,1)$ which satisfies

$$\Lambda_1^\sharp=N\Lambda_0^\sharp=\Lambda_0^\sharp\,{}^tN\,.$$

For any integer $k \geq 1$, we set

$$\Lambda_k^\sharp = N^k \Lambda_0 \sharp\,,$$

and we denote by Λ_k the 2-times contravariant, skew-symmetric tensor field associated with the vector bundle map $\Lambda_k^\sharp$. Let Z be a nowhere vanishing vector field on M such that

$$\mathcal{L}(Z)\Lambda_0 = 0\,, \qquad \mathcal{L}(Z)\Lambda_1 = 0\,.$$

It satisfies also

$$\mathcal{L}(Z)N = 0\,.$$

We assume that the set $\dot{M}$ of integral curves of Z is a smooth manifold and that the canonical projection $\pi : M \to \dot{M}$ is a submersion. We can easily prove that there exists on $\dot{M}$ a pair $(\dot{\Lambda}_0, \dot{\Lambda}_1)$ of Poisson structures, such that $\pi : M \to \dot{M}$ is a Poisson map from (M, Λ_0) onto $(\dot{M}, \dot{\Lambda}_0)$, and also from (M, Λ_1) onto $(\dot{M}, \dot{\Lambda}_1)$. Similarly, for any integer $k \geq 1$, there exists on $\dot{M}$ a 2-times contravariant, skew-symmetric tensor field $\dot{\Lambda}_k$, which is the direct image $\pi_*\Lambda_k$ of Λ_k by the map π.

4.4. Proposition. *With the same notations and assumptions as in 4.3, let $\mathcal{T}(N)$ be the Nijenhuis torsion of N. We assume that, for any pair (α, β) of Pfaff forms on M,*

$$\mathcal{T}(N)(\Lambda_1^\sharp\alpha, \Lambda_1^\sharp\beta) = 0\,.$$

Then for all $k \geq 1$, the Λ_k are Poisson tensors on M, and are pairwise compatible; in other words, for all $k \geq 1$ and $l \geq 1$, $[\Lambda_k, \Lambda_l] = 0$.

Proof. Let α be a Pfaff form and X a vector field on M. We use 3.4.(ii), with $P = \Lambda_0$ and $\beta = (\Lambda_0^\sharp)^{-1}X$. Since $[\Lambda_0, \Lambda_0] = 0$ and $[\Lambda_1, \Lambda_1] = 0$, we obtain

$$N\mathcal{R}(\Lambda_0, N)(\alpha, X) = \mathcal{T}(N)(\Lambda_0^\sharp\alpha, X)\,. \qquad (*)$$

Now we use 3.4.(iii) with $P = Q = \Lambda_1$. Since $[\Lambda_1, \Lambda_1] = 0$, we obtain for any pair (α, β) of Pfaff forms,

$$[\Lambda_2, \Lambda_1](\alpha, \beta) = -\Lambda_1\mathcal{R}^*(\Lambda_1, N)(\alpha, \beta) - \mathcal{R}(\Lambda_1, N)(\beta, \Lambda_1^\sharp\alpha) + \mathcal{R}(\Lambda_1, N)(\alpha, \Lambda_1^\sharp\beta)\,.$$

But by 3.4(i) and the formula $(*)$ above, since $\mathcal{T}(N)(X, Y) = -\mathcal{T}(N)(Y, X)$,

$$\mathcal{R}(\Lambda_1, N)(\alpha, X) = N\mathcal{R}(\Lambda_0, N)(\alpha, X) + \mathcal{T}(N)(X, \Lambda_0^\sharp\alpha) = 0\,.$$

Similarly,

$$\big\langle \mathcal{R}^*(\Lambda_1, N)(\alpha, \beta), X\big\rangle = \big\langle \beta, \mathcal{R}(\Lambda_1, N)(\alpha, X)\big\rangle = 0\,.$$

Therefore $[\Lambda_2, \Lambda_1] = 0$. Now we use Proposition 3.3, in which we replace Λ_0 by Λ_1 and Λ_1 by Λ_2, and we obtain the stated results. □

4.5. Remarks.

1. Under the assumptions of Proposition 4.4, for all $k \geq 1$, the images $\dot{\Lambda}_k = \pi_*\Lambda_k$ of the Poisson tensors Λ_k by the submersion π are Poisson tensors on $\dot{M}$, and they are pairwise compatible. This result remains true under assumptions slightly weaker than those made in that Proposition.

2. By using the formula indicated in 3.5, we can easily prove that the assumption of Proposition 4.4, that for any pair (α, β) of Pfaff forms on M,

$$\mathcal{T}(N)(\Lambda_1^\sharp \alpha, \Lambda_1^\sharp \beta) = 0 \,,$$

is equivalent to the following assumption: for any triple (α, β, γ) of Pfaff forms on M,

$$[\Lambda_1, \Lambda_0]({}^t\!N\alpha,\ {}^t\!N\beta,\ {}^t\!N\gamma) = 0 \,.$$

Proposition 4.4 does not say anything about the compatibility of the Λ_k with Λ_0, nor with the compatibility of the $\dot{\Lambda}_k$ with $\dot{\Lambda}_0$. The next Proposition deals with that question.

4.6. Proposition. *Under the assumptions of Proposition 4.4, we assume in addition that for any pair $(\dot\alpha, \dot\beta)$ of Pfaff forms on $\dot M$ and any vector field X on M,*

$$[\Lambda_1, \Lambda_0](\pi^*\dot\alpha, \pi^*\dot\beta) = 0 \,,$$
$$\mathcal{R}(\Lambda_0, N)(\pi^*\dot\alpha, X) = 0 \,,$$
$$\mathcal{T}(N)(X, \Lambda_1^\sharp \pi^*\dot\alpha) = 0 \,.$$

Then for any $k \geq 1$, the Poisson tensor $\dot\Lambda_k$ on $\dot M$ is compatible with $\dot\Lambda_0$.

Proof. Let $(\dot\alpha, \dot\beta)$ be a pair of Pfaff forms on $\dot M$. We have seen in the proof of 4.4 that $\mathcal{R}(\Lambda_1, N) = 0$. We can easily prove by induction on k that for all $k \geq 1$, first (by using 3.4.(i)) that $\mathcal{R}(\Lambda_k, N)(\pi^*\dot\alpha, X) = 0$, second (by using 3.4.(iii)) that $[\Lambda_k, \Lambda_0](\pi^*\dot\alpha, \pi^*\dot\beta) = 0$. This result implies that for all $k \geq 1$, $[\dot\Lambda_k, \dot\Lambda_0] = 0$. ∎

5. Application to the Toda lattice

5.1. The Toda lattice and the Flaschka transformation. On $\mathbf{R}^{2n}$ (coordinates $q^1, \ldots, q^n, p_1, \ldots, p_n$), let

$$\Lambda_0 = \sum_{i=1}^{n} \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q^i}$$

be the canonical Poisson tensor. The Toda lattice is the Hamiltonian system (relative to Λ_0) with Hamiltonian

$$H = \frac{1}{2} \sum_{i=1}^{n} {p_i}^2 + \sum_{i=1}^{n-1} \exp(q_i - q_{i+1}) \,.$$

We set, for $1 \leq i \leq n-1$, $1 \leq j \leq n$,

$$a^i = \frac{1}{2}\exp\left(\frac{1}{2}(q^i - q^{i+1})\right), \qquad b_j = -\frac{1}{2}p_j\,.$$

The map $\pi : \mathbf{R}^{2n} \to \mathbf{R}^{2n-1}$, $(q^1, \dots, q^n, p_1, \dots, p_n) \mapsto (a^1, \dots, a^{n-1}, b_1, \dots, b_n)$, is the Flaschka transformation [3]. The fibers of π are integral curves of the vector field

$$Z = \sum_{i=1}^{n} \frac{\partial}{\partial q^i} = \Lambda_0^\sharp\left(d\sum_{i=1}^{n} p_i\right).$$

Since $\mathcal{L}(Z)\Lambda_0 = 0$, there is on the image of π a unique Poisson tensor $\dot\Lambda_0$ such that π is a Poisson map. This Poisson tensor, which extends to the whole $\mathbf{R}^{2n-1}$, is

$$\dot\Lambda_0 = \frac{1}{4}\sum_{i=1}^{n-1} a^i \frac{\partial}{\partial a^i} \wedge \left(\frac{\partial}{\partial b_i} - \frac{\partial}{\partial b_{i+1}}\right).$$

The Hamiltonian H of the Toda lattice factors through π: it can be expressed as $H = \dot H \circ \pi$, with

$$\dot H = 2\sum_{i=1}^{n} {b_i}^2 + 4\sum_{i=1}^{n-1} (a^i)^2\,,$$

and the Toda vector fields projects by π onto the Hamiltonian vector field $T = \dot\Lambda_0^\sharp(d\dot H)$.

5.2. Higher order Poisson structures for the Toda lattice. The projection T of the Toda vector field on $\mathbf{R}^{2n-1}$ is Hamiltonian with respect to $\dot\Lambda_0$; it is Hamiltonian for another Poisson tensor $\dot\Lambda_1$, obtained by Adler [1]:

$$\begin{aligned}\dot\Lambda_1 = &-2\sum_{i=1}^{n-1}(a^i)^2 \frac{\partial}{\partial b_i} \wedge \frac{\partial}{\partial b_{i+1}} \\ &+ \frac{1}{2}\sum_{i=1}^{n-1} a^i \frac{\partial}{\partial a^i} \wedge \left(a^{i+1}\frac{\partial}{\partial a^{i+1}} + 2b_{i+1}\frac{\partial}{\partial b_{i+1}} - 2b_i\frac{\partial}{\partial b_i}\right),\end{aligned}$$

with, by convention, $a^n = 0$. We have indeed

$$T = \dot\Lambda_0^\sharp(d\dot H) = \dot\Lambda_1^\sharp(d\dot L)\,, \quad \text{with} \quad \dot L = \sum_{i=1}^{n} b_i\,.$$

Moreover, the Poisson tensors $\dot\Lambda_1$ and $\dot\Lambda_0$ are compatible: $[\dot\Lambda_1, \dot\Lambda_0] = 0$.

Kupershmidt [6] has found a third Poisson tensor $\dot\Lambda_2$ on $\mathbf{R}^{2n-1}$, whose coefficients are polynomials of degree 3 in the coordinates a^i and b_j, compatible with $\dot\Lambda_0$ and $\dot\Lambda_1$.

Damianou [2] has shown the existence of an infinite sequence $(\dot{\Lambda}_k)$, $k \in \mathbf{N}$, of Poisson tensors on $\mathbf{R}^{2n-1}$, the coefficients of $\dot{\Lambda}_k$ being polynomials of degree $k+1$ in the coordinates a^i, b_j, the first terms $\dot{\Lambda}_0$, $\dot{\Lambda}_1$, $\dot{\Lambda}_2$ of that sequence being the Poisson tensors already obtained by Flaschka, Adler and Kupershmidt. Moreover the terms of that sequence are pairwise compatible. These Poisson tensors are not obtained from each other by repeated application of a recursion operator, and Damianou does not indicate an explicit formula for obtaining them.

Such an explicit formula can be obtained by going back to $\mathbf{R}^{2n}$ and using Proposition 4.4. First we set

$$z = \frac{1}{n} \sum_{i=1}^{n} q^i .$$

Then $z, a^1, \ldots, a^{n-1}, b_1, \ldots, b_n$ can be used as (nonlinear) coordinates on $\mathbf{R}^{2n}$. The Poisson tensors $\dot{\Lambda}_0$ and $\dot{\Lambda}_1$ on $\mathbf{R}^{2n-1}$ can now be considered as Poisson tensors on $\mathbf{R}^{2n}$, whose expressions, in terms of these coordinates, are those given above, which do not contain z nor $\dfrac{\partial}{\partial z}$. We have

$$\Lambda_0 = \dot{\Lambda}_0 + Z \wedge Y ,$$

where Y is a vector field and Z the vector field already defined. Their expressions, in the new coordinates z, a^i, b_j, are

$$Y = \frac{1}{2n} \sum_{i=1}^{n} \frac{\partial}{\partial b_i} , \qquad Z = \frac{\partial}{\partial z} .$$

Although Λ_0 and $\dot{\Lambda}_1$, considered as Poisson tensors on $\mathbf{R}^{2n}$, are not compatible, we can define a recursion operator $N = \dot{\Lambda}_1^\sharp \circ (\Lambda_0^\sharp)^{-1}$, and set, for any integer $k \geq 1$, $\Lambda_k^\sharp = N^k \circ \Lambda_0^\sharp$. Let Λ_k be the corresponding 2-times contravariant skew-symmetric tensor fields.

By using 4.5.2, we can easily prove that the torsion $\mathcal{T}(N)$ of the recursion operator N satisfies the condition of Proposition 4.4. Therefore, the Λ_k, for $k \geq 1$, are pairwise compatible Poisson tensors. Their projections $\dot{\Lambda}_k$ by π are Poisson tensors on $\mathbf{R}^{2n-1}$; the first terms $\dot{\Lambda}_1$ and $\dot{\Lambda}_2$ are equal to those found by Adler and Kupershmidt. We conjecture that the whole sequence $(\dot{\Lambda}_k)$, $k \in \mathbf{N}$, obtained by that method, is the same as that obtained by Damianou (the non uniqueness of the terms in that sequence observed by Damianou corresponding to the various possible choices of a Poisson tensor Λ_1 on $\mathbf{R}^{2n}$ which projects by π onto $\dot{\Lambda}_1$).

6. References

[1] Adler, M., On a trace functional for formal pseudo-differential operators and the symplectic structure of the Korteweg de-Vries type equations, *Invent. Math.*, 50, 1979, 219–248.

[2] Damianou, P. A., Master symmetries and R-matrices for the Toda lattice, *Letters in Mathematical Physics*, 20, 1990, 101–112.

[3] Flaschka, H., On the Toda lattice, *Phys. Rev.* B 9, 1974, 1924–1925.

[4] Holm, D. D., and Marsden, J. E., The rotor and the pendulum, in *Symplectic geometry and mathematical physics*, (P. Donato et al., eds), Birkhäuser, Boston, 1991, 189–203.

[5] Koszul, J.-L., Crochet de Schouten-Nijenhuis et cohomologie, in *Élie Cartan et les mathématiques d'aujourd'hui*, Astérisque, numéro hors série, 1985, 257-271.

[6] Kupershmidt, B., Discrete Lax equatioons and differential difference calculus, *Astérisque* 123, 1985.

[7] Libermann, P., and Marle, C.-M., *Symplectic geometry and analytical mechanics*, D. Reidel Publishing Company, Dordrecht, 1987.

[8] Lichnerowicz, A., Les variétés de Poisson et leurs algèbres de Lie associées, *J. Differential Geometry*, 12, 1977, 253–300.

[9] Magri, F., and Morosi, C., A geometric characterization of integrable Hamiltonian systems through the theory of Poisson-Nijenhuis manifolds, *Quaderno S 19*, 1984, Università di Milano.

[10] Magri, F., Geometry and soliton equations, in *La "Mécanique analytique" de Lagrange et son héritage*, Acta Academiae Scientiarum Taurinensis, Torino 1990, 181–209.

[11] Marsden, J. E., and Ratiu, T., Reduction of Poisson manifolds, *Letters in Mathematical Physics*, 11, 1986, 161–169.

[12] Nijenhuis, A., X_{n+1}-forming sets of eigenvectors, *Proc. Kon. Ned. Akad. Wet. Amsterdam*, A 54, 1951, 200–212.

[13] Nijenhuis, A., Jacobi-type identities for bilinear differential concomitants of certain tensor fields I, *Indagationes Math.*, 17, 1955, 390–403.

[14] Ouzilou, R., Hamiltonian actions on Poisson manifolds, in *Symplectic geometry* (A. Crumeyrolle and J. Grifone, eds), Research notes in mathematics 80, Pitman, Boston, 1983, 172–183.

[15] Schouten, J. A., On the differential operators of the first order in tensor calculus, in *Convegno Int. Geom. Diff. Italia*, 1953. Ed. Cremonese, Roma, 1954, 1–7.

[16] Vaisman, I., *Lectures on the geometry of Poisson manifolds*, Birkhäuser, Boston, 1994.

[17] Weinstein, A., The local structure of Poisson manifolds, *J. Differential Geometry*, 18, 1983, 523–557 and 22, 1985, 255.