Various approaches to conservative and nonconservative nonholonomic systems

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March 1998

Abstract

We propose a geometric setting for the Hamiltonian description of mechanical systems with a nonholonomic constraint, which may be used for constraints of general type (nonlinear in the velocities, and such that the constraint forces may not obey Chetaev's rule). Such constraints may be realized by servomechanisms; therefore, the corresponding mechanical system may be nonconservative. In that setting, the kinematic properties of the constraint are described by a submanifold of the tangent bundle, mapped, by Legendre's transformation, onto a submanifold (called the Hamiltonian constraint submanifold) of the phase space (*i.e.*, of the cotangent bundle to the configuration manifold). The dynamical properties of the constraint are described by a vector subbundle of the tangent bundle to the phase space along the Hamiltonian constraint submanifold. In order to be able to deal with systems obtained by reduction by a symmetry group, we generalize that setting by using a Poisson structure on phase space, instead of the canonical symplectic structure of a cotangent bundle. The proposed geometric setting allows a very straightforward reduction procedure, which we compare with other reduction procedures, in particular that of Bates and Śniatycki [5]. Possible generalizations for systems with controlled kinematic constraints are briefly indicated.

1 Introduction

The theory of mechanical systems with kinematic constraints goes back to the last century, with important contributions by Hertz (1894), Ferrers (1871), Neumann (1888), Vierkandt (1892), Chaplygin (1897). See the references in the treatise by E. T. Whittaker [46], and in J. Herman's dissertation [21]. Several recent papers show a strong renewal of interest in that theory. They use various methods of modern differential geometry to obtain the equations of motion under an invariant form: connections [23, 36, 9], almost product structures [11, 18, 19], jet bundles [33, 34]; some of these papers consider the reduction by symmetry groups [23, 9, 5, 6, 13, 14, 20, 21], other establish relations with new developments in control theory and sub-riemannian geometry [9, 43].

In the present paper, we will discuss mainly the following two problems.

1. The problem of "unnatural" constraints. Natural kinematic constraints, realized for example by a solid body which rolls without slipping on another solid body, lead to restrictions of the set of admissible kinematic states which are linear (or, more generally, affine) in the velocities. For constraints of that type, d'Alembert's principle allows the determination of the set of possible values of the constraint forces from the set of admissible kinematic states only. Ideal constraints (also called perfect constraints) can be defined as constraints for which d'Alembert's principle is valid; therefore, for a mechanical system whose kinematic constraints are natural (linear or affine in the velocities) and ideal, the dynamical properties of the constraint are completely determined when one knows the set of admissible kinematic states. Nonlinear kinematic constraints were considered since the beginning of this century. Appell [1, 2] gave an example in which a constraint nonlinear in the velocities is obtained by a combination of constraints linear in the velocities, after neglecting the masses and inertia of some parts of the system and ignoring the corresponding degrees of freedom in the choice of the configuration space. The generalization of d'Alembert's principle for systems of that type is not obvious; in other words, what should be called an ideal constraint is not clear when the constraint is not linear, nor affine in the velocities. Appell proposed a way of generalizing d'Alembert's principle (which amounts to a local linearization, and will be described below), later more fully discussed by Chetaev [12] and, still more recently, in a different setting, by Vershik and Faddeev [42]. Very early, Delassus [16, 17] observed that the proposed generalization of d'Alembert's principle is not valid for systems for which the nonlinear constraint is obtained by a limit process, as in Appell's example. The development of sensors, computers, servomechanisms and other high technology devices makes easy now the realization of mechanical systems with constraints nonlinear in the velocities. It is therefore important to have an unambiguous way of writing down valid equations of motion for such systems. We will show that a slight modification of the procedure used in [30] leads to such equations. Instead of deducing the space of possible values of the constraint forces from the space of admissible kinematic states, as done by applying d'Alembert's principle, we have only to consider that these two spaces are independently given and describe, respectively, the dynamical and the kinematical properties of the constraint. These two separate sets of data must of course satisfy some compatibility conditions. This point was frequently overlooked in the recent literature, even by the present author in [30]; however, it was clearly seen by other authors, for example Dazord [15]; we emphasized it in [31].

2. The reduction problem. In [30] we developed a geometric construction, adapted from the works of Vershik and Faddeev [41, 42, 43], which leads to the equations of motion of a constrained mechanical system in a coordinate-free, intrinsic form. That construction, eventually modified, as indicated above, to deal with general constraints, is in good agreement with the reduction procedure for mechanical systems with symmetry: the equations of motion of the reduced system are obtained by application of the same procedure as that used for the original system, the only change being the use of a Poisson structure on the reduced phase space, instead of a symplectic structure. Other reduction procedures for mechanical systems with nonholonomic constraints were proposed (Koiller [23], Bates and Śniatycki [5], Bloch, Krishnaprasad, Marsden and Murray [9], Koon and Marsden [24, 25]. In the following, we will compare our approach with these different approaches to reduction.

For simplicity, we will consider below only time-independent constraints. The main ideas presented here should be easily transposed for systems with time-dependent constraints, when the variation of the constraints as a function of time is given. Massa and Pagani [34] have recently presented a very thorough discussion of mechanical systems with time dependent nonholonomic constraints. Generalizations for systems with controlled kinematic constraints, (that means, kinematic constraints which may vary with time, but in a way not given in advance, which may be chosen in order to control the behaviour of the system) will be briefly discussed in the final section.

2 Mechanical systems with additional forces

2.1 Lagrange's formalism

We consider a mechanial system, for which the set of all possible positions of its parts, at a given time, is a smooth manifold Q. That manifold will be called the *configuration space* of the system. The set of all possible positions and velocities of the parts of the system, at a given time, is then the tangent bundle TQ, which will be called the space of *kinematic states* of the system.

We will assume that the dynamical properties of the system are mathematically described by a smooth function $L : TQ \to \mathbf{R}$, called the *Lagrangian* of the system. That function determines a map $\Delta(L)$, called the *Lagrange differential* of L, defined on the set $J^2(\mathbf{R}, 0, Q)$ of second order jets of $\mathbf{R}$ into Q at the origin, with values in the cotangent bundle T^*Q, fibered over Q. That map may be defined in an intrinsic way [37, 39]. Let us simply recall its expression in local coordinates. We use a chart of Q in which the local coordinates are $(x^i,\ 1 \leq i \leq n)$, and we denote by (x^i, v^i) and (x^i, p_i) the local coordinates in the associated charts of TQ and T^*Q. Let $c : \mathbf{R} \to Q$ be a parametrized C^2 curve in Q. We denote by $(x^i(t), v^i(t))$ the local coordinates of $\dfrac{dc(t)}{dt}$. Then the coordinates of $\Delta(L)(j^2c(t_0))$ are $(x^i(t), \pi_i(t))$, with

$$\pi_i(t) = \sum_{i=1}^{n} \left(\frac{d}{dt} \left(\frac{\partial L(x(t), v(t))}{\partial v^i} \right) - \frac{\partial L(x(t), v(t))}{\partial x^i} \right) \Bigg|_{t=t_0} .$$

We will assume that some additional forces, not already accounted for by the Lagrangian L, are acting on the system. These forces will be specified in the next section: they will be the constraint forces. When the system's configuration is a point $x \in Q$, these additional forces are mathematically described by an element f (in general unknown) of the cotangent space T^*_xQ. The equations of motion of the system, in Lagrange's formalism, can be written in coordinate-free form, as

$$\Delta(L)(j^2c(t)) = f\,, \tag{1}$$

or, in local coordinates,

$$\frac{d}{dt} \left(\frac{\partial L(x, v)}{\partial v^i} \right) - \frac{\partial L(x, v)}{\partial x^i} = f_i\,, \qquad 1 \leq i \leq n\,. \tag{2}$$

2.2 Hamilton's formalism

The Lagrangian L determines, in a unique way, a smooth map $\mathcal{L} : TQ \to T^*Q$, fibered over Q, called the *Legendre transformation*. We refer to Tulczyjew [38] for an intrinsic very general description of that transformation, and recall here that if we use, as above, local coordinates (x^i, v^i) on TQ and (x^i, p_i) on T^*Q, the Legendre transformation is expressed as

$$\mathcal{L} : (x^i,\ v^i) \mapsto \left(x^i,\ p_i = \frac{\partial L(x,v)}{\partial v^i}\right) .$$

We will assume in the following that the Lagrangian L is regular, *i.e.*, that the Legendre transformation $\mathcal{L}$ is a diffeomorphism of TQ onto T^*Q. The cotangent bundle T^*Q will be called the *phase space* of the mechanical system, and points in T^*Q will be called *dynamical states* of that system.

A motion, given by a parametrized smooth curve $t \mapsto c(t)$ in Q, will be represented, in Hamilton's formalism, by the parametrized curve $t \mapsto \tilde{c}(t) = \mathcal{L}\left(\frac{dc(t)}{dt}\right)$ in T^*Q. We introduce the Hamiltonian $H : T^*Q \to \mathbf{R}$, given by $H = (i(Z)L - L) \circ \mathcal{L}^{-1}$, where Z is the Liouville vector field on TQ. We introduce also the Liouville 1-form α, the canonical symplectic 2-form $\Omega = d\alpha$ on T^*Q, and the Hamiltonian vector field X_H associated with H, defined by $i(X_H)\Omega = -dH$.

A final ingredient is needed in order to write down the equations of motion in Hamilton's formalism: an intrinsic way of introducing the additional forces. For that purpose we define a map $\lambda : T^*Q \times_Q T^*Q \to TT^*Q$, as follows: for $x \in Q$, ξ and $\eta \in T^*_xQ$, $\lambda(\xi, \eta)$ is the vector, element of $T_\xi(T^*Q)$, tangent at ξ to the fibre T^*_xQ, and equal to η (the tangent space at ξ to the vector space T^*_xQ being identified with that vector space). We observe that $\lambda(\xi, \eta)$ is vertical: that means that if we denote by $q : T^*Q \to Q$ the canonical projection and $Tq : TT^*Q \to TQ$ its canonical lift to vectors, then $Tq(\lambda(\xi, \eta)) = 0$.

The map λ can be interpreted in terms of the canonical symplectic 2-form Ω of T^*Q. For $\xi \in T^*_xQ$ fixed, the inverse of the map $\eta \mapsto \lambda(\xi, \eta)$ is obtained by composing the pullback $q^* : T^*_xQ \to T^*_\xi(T^*Q)$ and the interior product with $\Omega(\xi)$:

$$i(\lambda(\xi, \eta))\Omega(\xi) = q^*\eta .$$

In local coordinates, we have indeed

$$\eta = \sum_i \eta_i dx^i , \quad \Omega = \sum_i dp_i \wedge dx^i , \quad \lambda(\xi, \eta) = \sum_i \eta_i \frac{\partial}{\partial p_i} .$$

Let us assume that the dynamical state of the system is a point $\xi \in T^*Q$, the corresponding configuration being $x = q(\xi) \in Q$, and that the additional forces are represented, in Lagrange's formalism, by $f \in T^*_xQ$. Then these additional forces will be represented, in Hamilton's formalism, by the vertical vector $\lambda(\xi, f)$, tangent to T^*Q at point ξ.

In Hamilton's formalism, the equations of motion are the well known Hamilton's equations, modified by introduction of the additional forces. They can be written, in coordinate-free form, as

$$\frac{d\tilde{c}(t)}{dt} = X_H(\tilde{c}(t)) + \lambda(\tilde{c}(t), f) , \tag{3}$$

or, in local coordinates,

$$\left\{ \begin{aligned} \frac{dx^i}{dt} &= \frac{\partial H(x,p)}{\partial p_i} , \\ \frac{dp_i}{dt} &= -\frac{\partial H(x,p)}{\partial x^i} + f_i , \end{aligned} \right. \qquad 1 \le i \le n . \tag{4}$$

3 Mechanical systems with constraints

The mechanical system under consideration is said to be *constrained*, or *submitted to kinematic constraints*, when the set of its possible kinematic states is a subset C of TQ, rather than the whole tangent bundle TQ. Such a limitation of the set of possible kinematic states is generally due to interactions of some parts of the system, either between themselves, or with external objects, not already accounted for by the definition of the configuration space Q. These interactions give rise to additional forces, not already accounted for by the Lagrangian L of the system. These forces, which will be called the *constraint forces*, are the additional forces introduced in the previous section, which were left unspecified up to now.

We shall assume in the following that the subset C of possible kinematic states is a smooth submanifold of TQ, called the *constraint submanifold*. Since the Legendre transformation $\mathcal{L}$ is a diffeomorphism of TQ onto T^*Q, the image $\mathcal{L}(C)$ of the constraint submanifold by the Legendre transformation is a smooth submanifold D of T^*Q, which will be called the *Hamiltonian constraint submanifold*. Let us observe that some authors, for example Weber [44], define a Hamiltonian constrained system by means of a distribution on the phase space T^*Q, instead of by means of a submanifold. We think that for applications to mechanical systems, the use of a submanifold is more natural; it may be seen as a particular case of Weber's formalism. As we will see later, the more general Weber's formalism could be useful for controlled constrained mechanical systems.

In Lagrange's formalism, the motion of the mechanical system must satisfy Lagrange's equations (1) and the constraint equation

$$\frac{dc(t)}{dt} \in C \quad \text{for all } t\,. \tag{5}$$

Equivalently, in Hamilton's formalism, the motion of the mechanical system must satisfy Hamilton's equations (3), and the constraint equation

$$\widetilde{c}(t) \in D \quad \text{for all } t\,. \tag{6}$$

When the constraint force f, which appears in Equations (1) or (3), is considered as unknown, the system made by (1) and (5) (or by (3) and (6)) is underdetermined. The additional equations which are needed must express physical properties of the constraint. As we shall see in the following, these additional equations generally restrict the space of possible values of the constraint force.

3.1 Classical kinematic constraints

A kinematic constraint will be called *classical* when it is linear, or more generally affine, in the velocities, *i.e.*, when the corresponding constraint submanifold C is an affine sub-bundle of a tangent bundle TQ_1, where Q_1 is a smooth submanifold of the configuration manifold Q.

Simplest examples of classical constraints are the *geometric constraints*, in which the restriction bears upon the set of possible configurations of the system, rather than on the set of possible kinematic states: the set of possible configurations is a smooth submanifold Q_1 of Q, and the constraint submanifold is the tangent bundle TQ_1, considered as a submanifold of TQ.

In more general classical kinematic constraints, the restriction bears upon possible velocities of parts of the system, as well as on possible configurations. Such constraints are encountered, in particular, in systems made of several rigid bodies, some of which are rolling without slipping on each other, or on other rigid bodies which do not belong to the system and whose motions are known and stationary.

3.1.1 Example

Let us consider a system made of a rigid three-dimensional heavy body. Let E be the physical space, G the group of Euclidean displacements of E, and P_0 a particular position of the rigid body in E. By associating with any position P of the body, the unique $g \in G$ such that $P = gP_0$, we can identify the configuration space Q of the system with the group G. Choosing an origin O in E allows us to consider E as an Euclidean three-dimensional vector space and to identify G with the semi-direct product $E \times \mathbf{SO}(E)$, an element (a, g) of that semi-direct product corresponding to the mapping of E into itself $x \mapsto a + gx$. We identify the Lie algebra $\mathcal{G}$ of G with $E \times \mathbf{so}(E)$, and the tangent bundle TG with $G \times \mathcal{G} = E \times \mathbf{SO}(E) \times E \times \mathbf{so}(E)$, by left translations.

Let us now assume that the rigid body is bounded by a smooth strictly convex surface and that it is supported by a fixed solid horizontal plane F. Moreover, let us assume that during its motion, the body remains in contact with the horizontal plane and that it rolls without slipping on it. We choose the origin O in the horizontal plane F, and the unit vector e_3 vertical and directed upwards. We denote by Σ the surface of the rigid body when it is in its reference position P_0, and by $\Gamma : S^2 \to \Sigma$ the inverse of the Gauss map of Σ. Since we assumed that the body is strictly convex, and that its boundary Σ is smooth, the Gauss map is a diffeomorphism of Σ onto S^2, so its inverse Γ is well defined. Then the constraint submanifold is the set of kinematic states (a, g, v, X) which satisfy

$$(a|e_3) = (\Gamma \circ g^{-1}(-e_3) \mid g^{-1}(-e_3)) \,, \tag{7}$$

$$v + X \circ \Gamma \circ g^{-1}(-e_3) = 0 \,, \tag{8}$$

where $(\ |\ )$ denotes the Euclidean scalar product in E (we refer to [30] for more details). The set of $(a, g) \in G$ which satisfy Equation (7) is a submanifold Q_1 of G which may be identified with $F \times \mathbf{SO}(E)$. Then Equation (8) shows that the constraint submanifold C is a (non integrable) vector sub-bundle of TQ_1.

3.1.2 Example

Let us now assume that the plane on which the body rests rotates around the vertical axis through the origin, at a given constant angular velocity ω. The constraint submanifold C is now the set of kinematic states (a, g, v, X) which satisfy Equation (7) and

$$g(v + X \circ \Gamma \circ g^{-1}(-e_3)) = \omega(a + g \circ \Gamma \circ g^{-1}(-e_3)) \,, \tag{9}$$

instead of (8). Clearly, C is now an affine sub-bundle of TQ_1, whose associated vector sub-bundle is the sub-bundle obtained previously as constraint submanifold when the plane F was assumed to be fixed, defined by Equation (8).

3.2 D'Alembert's principle and ideal classical kinematic constraints

We now come to the description of constraint forces for a classical kinematic constraint. Let C be the constraint submanifold. According to the definition given in the previous section, C is an affine sub-bundle of a tangent bundle TQ_1, where Q_1 is a smooth submanifold of Q. For any $x \in Q_1$, we denote by C_x the fibre $C \cap T_xQ$ of C over x; therefore C_x is an affine subspace of T_xQ_1, which is itself a vector subspace of T_xQ, so we can consider C_x as an affine subspace of T_xQ. We shall denote by $\overrightarrow{C_x}$ the vector subspace of T_xQ associated with C_x, and by $(\overrightarrow{C_x})^0$ its annihilator, *i.e.*, the vector subspace of elements η in T^*_xQ such that, for all $v \in \overrightarrow{C_x}$, $\langle \eta, v \rangle = 0$. When x runs over Q_1, we obtain a vector sub-bundle $\overrightarrow{C}$ of $T_{Q_1}Q$, whose fibre at x is $\overrightarrow{C_x}$, called the bundle of *admissible infinitesimal virtual displacements*. Its annihilator $(\overrightarrow{C})^0$ is the vector sub-bundle of $T^*_{Q_1}Q$ whose fibre at x is $(\overrightarrow{C_x})^0$.

According to the so called d'Alembert's principle, the classical constraint is said to be *ideal* if the infinitesimal work of the constraint force vanishes for any admissible infinitesimal virtual displacement. This amounts to say that the constraint force takes its value in $(\overrightarrow{C})^0$.

In Hamilton's formalism, d'Alembert's principle states that for each admissible dynamical state $\xi \in D$, the constraint force (which is an element of $T_\xi(T^*Q)$) must belong to the subspace $W_\xi = \lambda(\xi, (\overrightarrow{C})^0)$. Using the expression of the map λ given in section 2, we obtain the following equivalent statement: the constraint force must belong to the vector subspace of $T_\xi(T^*Q)$ which is the symplectic orthogonal of $(Tq)^{-1}(\overrightarrow{C})$, with respect to the canonical symplectic 2-form of T^*Q. Indeed, let $\xi \in D$, and $x = q(\xi) \in Q$. A vector $w \in T_\xi(T^*Q)$ belongs to $W_\xi = \lambda(\xi, (\overrightarrow{C})^0)$ if and only if $i(w)\Omega(\xi)$ belongs to $q^*((\overrightarrow{C_x})^0)$, that means, if and only if $\Omega(\xi)(w, v) = 0$ for all $v \in T_\xi(T^*Q)$ such that $Tq(v) \in (\overrightarrow{C_x})^0$. We therefore have

$$W_\xi = \operatorname{orth}_{\Omega(\xi)} Tq^{-1}(\overrightarrow{C_x}) \,,$$

where, for any vector subspace F of $T_\xi(T^*Q)$, $\operatorname{orth}_{\Omega(\xi)}F$ denotes the symplectic orthogonal of F with respect to $\Omega(\xi)$.

3.2.1 Remark

The reader should be warned that some authors use a diffferent definition of ideal constraints; for example, de León, Marrero and de Diego [19] consider that a constraint (which may be nonlinear in the velocities) is ideal when the constraint submanifold C is tangent to the Liouville vector field of the vector bundle TQ. The reason for that language convention is that when the constraint forces for such a constraint are definied by application of Chetaev's rule (to be discussed below), the total energy of the system remains constant during the motion. Carathéodory observed that property for homogeneous constraints in 1934 (see Benenti [7]). Therefore a constraint defined by an affine sub-bundle of TQ is not ideal in de León's terminology, but may be ideal in our terminology.

3.2.2 A property of ideal classical constraints for normal Lagrangians

Let $v \in TQ$ be a kinematic state, $x = p(v) \in Q$ be the corresponding configuration, and L_x be the restriction of the Lagrangian L to the tangent space T_xQ. We will say that the Lagrangian L is *normal* at v if the matrix

$$\left(\frac{\partial^2 L_x}{\partial v^i \partial v^j}(v) \right) , \qquad 1 \leq i, j \leq n \,,$$

is positive definite. We have denoted by $v^1, \dots, v^n$ the coordinates in T_xQ associated with a basis of that space. This definition has an intrinsic meaning, since the positive definiteness of that matrix does not depend on the choice of the basis of T_xQ.

D'Alembert's principle has the following nice mathematical property: for a mechanical system whose Lagrangian is everywhere normal, with an ideal classical kinematic constraint, the restriction of the space of constraint forces obtained by application of that principle, added to Equations (1) and (5) in Lagrange's formalism (or Equations (3) and (6) in Hamilton's formalism), leads to a well behaved system of differential equations (see for example [30]). We shall illustrate that property for the simple example of a geometric constraint, where $C = TQ_1$ (the same property holds for a general kinematic constraint defined by an affine sub-bundle of TQ_1). In that case, an additional simplification occurs: the geometric constraint can be eliminated by using Q_1 instead of Q as configuration space and $L_1 = L|_{TQ_1}$ instead of L as Lagrangian. Let us indeed use a chart of Q adapted to the submanifold Q_1, in which the local coordinates $(x^i,\ 1 \leq i \leq n)$ are such that Q_1 is locally defined by the equations $x^{p+1} = 0,\ \dots,\ x^n = 0$. The equations of motion of the system, in Lagrange's formalism, are

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L(x,v)}{\partial v^i}\right) - \frac{\partial L(x,v)}{\partial x^i} &= 0\,, \qquad 1 \leq i \leq p\,, &(10)\\ \frac{d}{dt}\left(\frac{\partial L(x,v)}{\partial v^j}\right) - \frac{\partial L(x,v)}{\partial x^j} &= f_j\,, \qquad p+1 \leq j \leq n\,, &(11)\\ x^j &= 0\,, \qquad p+1 \leq j \leq n\,. &(12) \end{aligned}$$

Denoting by $y = (y^1, \dots, y^p)$ the local coordinates on Q_1 and by $(y,w) = (y^1, \dots, y^p, w^1, \dots, w^p)$ the associated local coordinates on TQ_1, we see that Equations (10) and (12) imply

$$\frac{d}{dt}\left(\frac{\partial L_1(y,w)}{\partial w^i}\right) - \frac{\partial L_1(y,w)}{\partial y^i} = 0\,, \qquad 1 \leq i \leq p\,,$$

which are simply the Euler-Lagrange's equations for the Lagrangian L_1 on TQ_1. Equations (11) need not be used to determine the motion; they just yield the value of the constraint force.

That elimination of the geometric constraint can be observed also in Hamilton's formalism: assuming that both L and its restriction L_1 are regular, and denoting by $\mathcal{L}_1 : TQ_1 \to T^*Q_1$ the Legendre transformation determined by L_1, we see that $\mathcal{L} \circ \mathcal{L}_1^{-1}$ is a diffeomorphism of T^*Q_1 onto D, and that the equations of motion on D imply the usual Hamilton's equations, for the unconstrained Hamiltonian system on T^*Q_1 whose Hamiltonian is $H_1 = H \circ \mathcal{L} \circ \mathcal{L}_1^{-1}$.

3.2.3 Validity of d'Alembert's principle

Several authors have questioned the validity of d'Alembert's principle for real mechanical systems. In particular, Arnol'd, Kozlov and Neishtadt [3] proposed another way to derive equations of motion for a constrained Lagrangian system, by using a conditional form of Hamilton's variational principle. The equations of motion so obtained differ from the equations obtained by application of d'Alembert's principle, by the presence of additional terms. However, for mechanical systems with a holonomic kinematic constraint (that means, when the vector sub-bundle $\overrightarrow{C}$ of TQ_1 is involutive), these two sets of equations coincide.

The method of derivation of the equations of motion proposed by Arnol'd, Kozlov and Neishtadt seems to be related with the following procedure, which was used by many authors (including P. Appell) at the end of the last century. That procedure amounts to what follows: first express some of the components (say, for example, the last $n-p$ components, for a given choice of local coordinates) of the generalized velocity $\dfrac{dc(t)}{dt}$, in terms of the other components (the first p components); then replace these components by their expressions in the Lagrangian L; and finally, write down the usual Lagrange's equations for that modified Lagrangian, considered as a function of $c(t)$ and of some of the components (the first p components) of $\dfrac{dc(t)}{dt}$.

As early as 1899, Korteweg [26] clearly pointed out that such a procedure leads to erroneous results, except for the study of small motions around an equilibrium. He even wrote that this fact was already stated earlier (1892) by Vierkandt. Of course, Arnol'd, Kozlov and Neishtadt variational method is not so crude as the procedure described above; in particular, contrary to that procedure, it has a clear intrinsic meaning, independent of the choice of a particular chart. However, we believe that real nonholonomic constrained systems such as those made by a solid body which rolls without slipping on another solid body, when the friction is low enough, are better described by equations obtained by application of d'Alembert's principle than by equations obtained by Arnol'd, Kozlov and Neishtadt variational method. This belief is strongly supported by the results of Lewis and Murray [27], who have made a thorough comparison of these two different systems of equations of motion, and have fully treated the example of a spherical solid body rolling on a rotating table. Their work even includes numerical calculations which take into account various friction forces, and comparison with experimental results. They conclude that the best agreement with experimental results is obtained with the equations derived by using d'Alembert's principle, modified by the inclusion of appropriate friction forces. Another recent comparison of equations obtained by application of d'Alembert's principle with equations obtained by Arnol'd, Kozlov and Neishtadt variational method can be found in the work of Cardin and Favretti [10].

3.3 Non classical kinematic constraints

Mechanical systems with kinematic constraints nonlinear in the velocities were considered very early (P. Appell [1], 1911). A very abundant literature, extending up to now, deals with various methods to obtain their equations of motion: application of Gauss' principle of least curvature, conditional Hamilton's variational principle, Hamilton-Jacobi equation, ... For a recent discussion of Gauss' principle, see [28]. The results are similar to those described in the previous section for classical kinematic constraints: different methods may lead to non equivalent sets of equations of motion. The difficulty encountered with such systems is the following. We have seen in Section 2 that when the constraint forces are considered as unknown, the system made by Lagrange's equations with additional forces (1) and by the constraint equations (5) (or, in Hamilton's formalism, by Hamilton's equations with additional forces (3) and the constraint equations (6)) is underdetermined. One may think that some physical properties of the constraints should impose restrictions to the set of possible values of the constraint forces, in such a way that the system of equations will become well behaved. It is exactly what occurs for classical ideal constraints, with the restrictions imposed by d'Alembert's principle. But it is not at all clear what should be defined as an ideal constraint, when that constraint is non classical (*i.e.*, neither linear nor affine in the velocities).

Around 1930, Chetaev [12] proposed a rule to restrict the set of possible values of the constraint forces. Non classical constraints which obey that rule (discussed, for example, by

Pironneau [35]) are known, at least in the Russian literature, as constraints of Chetaev's type; let us observe, however, that equations written earlier by Appell in [1] are in fact precisely those one would obtain by application of Chetaev's rule! Expressed in an intrinsic form, that rule is the following. Let C be the constraint submanifold, $v \in C$ a kinematic state, and $x = p(v)$ the corresponding configuration (we have denoted by $p : TQ \to Q$ the canonical projection). The fibre $C_x = C \cap T_xQ$ is assumed to be smooth, so there exists a tangent space T_vC_x to it at point v. Since T_xQ is a vector space, T_vC_x can be considered as one of its vector subspaces. When the kinematic state of the system is v, Chetaev's rule states that the space of possible values of the constraint force is the annihilator of T_vC_x. Clearly, when the constraint is classical, C_x is an affine subspace of T_xQ and Chetaev's rule is equivalent to d'Alembert's principle. That rule was introduced more recently in a different setting by Vershik and Faddeev [41, 42, 43], and used in [30] as a definition of ideal non classical constraints, where we have shown that under some rather mild assumptions, it leads to a well behaved system of differential equations for the motion of the system.

Application of Chetaev's rule may lead to wrong results. This can be seen on the famous example, first considered by Appell [2], of a heavy material point moving in space, with the kinematic constraint

$$v_x^2 + v_y^2 = a^2 v_z^2 ,$$

where a is a constant, and v_x, v_y and v_z the components of the velocity of the moving point in a Cartesian frame with a vertical z axis. Applying Chetaev's rule, we can easily see that the trajectories of the moving point are straight lines. However, Appell [2] described a way in which that kinematic constraint can be realized in practice, by using a wheel rolling without sliding on a horizontal plane, a thread wounded on a smaller coaxial wheel, and skates sliding without friction on the plane. Appell's machine is described also in [35] and [22]. A careful analysis of that machine shows that its configuration space is at least 4-dimensional (while the configuration space of the material point alone, when ignoring the accessories used for the practical realization of the constraint, is 3-dimensional), and that the motion of the moving point submitted to that kinematic constraint by means of Appell's machine is a spiral rather than a straight line. Delassus [16, 17] discussed further these results, by considering a mechanical system S coupled with another mechanical system S_1, the total system being submitted to classical constraints (linear or affine in the velocities), in such a way that when one eliminates the coordinates of S_1 and their time derivatives, one obtains for S alone a non classical kinematic constraint. He proves that when the masses of all the parts of S_1 vanish, the equations of motion of S do not converge towards the equations which would be obtained by applying Chetaev's rule to the non classical constraint of S, considered alone. Of course, at that time (1911), Chetaev's rule was not yet called by that name.

Another example of realization of a constraint nonlinear in the velocities is given by Benenti [7], for a system of two material points moving in a plane, their velocities being constrained to remain always parallel to each other. As for Appell's machine, the realization of that constraint is obtained by means of an auxiliary system of four rigid rods whose masses and inertia are neglected. Delassus' results apply to that sysem, as well as to Appell's machine.

Practical realization of non classical kinematic constraints by means of wheels, skates, and similar mechanical components is rather difficult (although not impossible, as shown by Appell's example). By using servomechanisms, such a realization becomes much easier, at least conceptually. A very simple example is a car moving on a road of varying slope, with a cruise control which keeps the speed constant. A more elaborate example, fully discussed by Bloch, Krishnaprasad, Marsden and Sánchez de Alvarez [8], is a rigid body with three rotors, acted upon by feedback devices, in order to stabilize its position. These authors obtain a remarkable result: although the total energy of the system is no more a constant of the motion, for a particular feedback law, the motion of the system is Hamiltonian with respect to the Lie-Poisson structure on the dual of the Lie algebra $\mathbf{so}(3)$. Such a result should be interpreted in terms of the concept of Hamiltonian constraint introduced by Dazord [15].

Let us look at simple examples, which show that Chetaev's rule cannot be used in general.

3.3.1 Example

This example is an idealization of the game in which one tries to keep a straight rod in equilibrium on the tip of one finger. For simplicity we assume that the rod remains in a fixed vertical plane and that the tip of the finger which supports it can move only along a straight horizontal line contained in that plane. We take that line as axis of coordinates Ox, the other axis Oz being vertical. The rod is free to rotate around its point of contact with the Ox axis. The configuration space is therefore $Q = \mathbf{R} \times S^1$, with coordinates (x, θ), where $x \in \mathbf{R}$ is the abscissa of the point of contact of the rod with the horizontal axis, and $\theta \in S^1$ the angle made by the rod with that axis. We assume that the strategy used by the player to keep the rod in equilibrium amounts to impose the value of $\dfrac{dx}{dt}$ as a function of x, θ and $\dfrac{d\theta}{dt}$. Such a strategy could be realized also with a servomechanism, instead of a human player. The constraint submanifold $C \subset TQ$ is therefore

$$C = \{\, (x, \theta, \dot{x}, \dot{\theta}) \mid \dot{x} = f(x, \theta, \dot{\theta}) \,\}\,,$$

where f is a known smooth function.

If the rotation of the rod around its point of contact with Ox is frictionless, the constraint force must belong to the annihilator of the vector sub-bundle of TQ generated by the vector field $\dfrac{\partial}{\partial\theta}$. Clearly, that result has nothing to do with Chetaev's rule.

Calculations show that the equilibrium position where the rod is vertical and rests on its lower end, which is unstable when no kinematic constraint is applied, becomes stable with a kinematic constraint in which $\dfrac{dx}{dt}$ is imposed as a suitable function of θ and $\dfrac{d\theta}{dt}$. For example, we may choose

$$\frac{m(I + ml^2\cos^2\theta)}{I + ml^2}\,\frac{dx}{dt} = a\left(\theta - \frac{\pi}{2}\right) - \left(\frac{I + ml^2\cos^2\theta}{l(I + ml^2)} + \beta\right) p_\theta\,,$$

where m is the mass of the rod, l the distance between the point of the rod on the Ox axis and its centre of mass, I its moment of inertia with respect to its centre of mass, a and β two constants wit $a < 0$ and $\beta > 0$, and

$$p_\theta = -ml\sin\theta\frac{dx}{dt} + (I + ml^2)\frac{d\theta}{dt}\,.$$

3.3.2 Example

Let us change slightly the previous example. We assume now that the servomechanism acts upon the angle θ of the rod with the horizontal axis, and that the point of contact of the rod with that axis can slide freely without friction. For simplicity we assume that the relation which links x, θ, $\frac{dx}{dt}$ and $\frac{d\theta}{dt}$ in the previous example can be solved with respect to $\frac{d\theta}{dt}$, as well as with respect to $\frac{dx}{dt}$. Then the constraint submanifold C will be the same as that in the previous example. But the constraint force must now belong to the annihilator of the vector sub-bundle of TQ generated by the vector field $\frac{\partial}{\partial x}$.

These two examples show that for non classical constraints realized by means of servomechnisms, Chetaev's rule cannot be used, and that the set of possible values of the constraint force is not determined by the constraint submanifold.

3.4 A general setting for constrained Hamiltonian systems

For the treatment of mechanical systems with constraints, we have already introduced several ingredients: the configuration space Q, the space of kinematic states TQ, a submanifold C of TQ called the constraint submanifold, the Lagrangian $L : TQ \to \mathbf{R}$. Assuming that the Lagrangian is regular, we may use equivalently Hamilton's formalism, in which the ingredients are the phase space T^*Q, the Hamiltonian constraint submanifold $D = \mathcal{L}(C) \subset T^*Q$, and the Hamiltonian $H : T^*Q \to \mathbf{R}$. For simplicity, we will consider in the following only Hamilton's formalism. In order to be able to deal with arbitrary constraints, including those realized by means of servomechanisms, we propose to introduce an additional ingredient: a vector sub-bundle W of $T_D(T^*Q)$, contained in the vertical sub-bundle $\ker(Tq)$, which will be the space of possible values of the constraint forces (in Hamilton's formalism).

For a system with an ideal classical kinematic constraint, obeying d'Alembert's principle, the bundle W has, for fibre at a point $\xi \in D$,

$$W_\xi = \mathrm{orth}_{\Omega(\xi)} Tq^{-1}(\overrightarrow{C_x}) \,,$$

as we have seen in section 3.2. A similar formula holds for a constraint nonlinear in the velocities obeying Chetaev's rule: we have only to replace C_x by $T_v C_x$, with the notations of Section 3.3, v being the kinematic state mapped onto the dynamical state ξ by the Legendre transformation.

More generally, to deal with systems obtained by reduction, we proposed [30] to replace the cotangent bundle T^*Q by a Poisson manifold (P, Λ) (see [29] and [45] for the definition and properties of Poisson manifolds). The Hamiltonian constraint submanifold is then a submanifold D of P and the set of possible values of the constraint forces a vector sub-bundle W of $T_D P$. The multiplet (P, Λ, H, D, W) will be called a *constrained Hamiltonian system*. Observe that our definition of that concept is not equivalent with that of Weber [44].

The constrained Hamiltonian system (P, Λ, H, D, W) will be called *regular* if $TD \cap W = \{0\}$ (the zero sub-bundle of $T_D P$), and if the restriction to D of the Hamiltonian vector field X_H is a section of the direct sum $TD \oplus W$. A similar regularity condition was introduced by Vershik and Faddeev [41, 42] under a slightly different form, and appears also in [18] and [19]. When it is satisfied, we have a well behaved system of differential equations as equations of motion, since the restriction to D of the Hamiltonian vector field X_H splits into a sum

$$X_H|_D = X_D + X_W \tag{13}$$

of a vector field X_D tangent to D (whose integral curves are the motions of the mechanical system), and a section X_W of W (which is the opposite of the set of values of the constraint forces).

A sufficient condition of regularity is given in [30], where we have shown that a mechanical system whose Hamiltonian is normal, with a noholonomic constraint (maybe nonlinear in the velocities) which obeys d'Alembert's principle, or more generally Chetaev's rule, is regular.

Non regular constrained systems obey implicit differential equations; they are considered in [22], [4] and [32].

3.4.1 Remark

Let us consider a constrained mechanical system whose configuration space is a manifold Q, with a regular Lagrangian $L : TQ \to \mathbf{R}$. Then the corresponding constrained Hamiltonian system (P, Λ, H, D, W) is not of the most general type. First, the Poisson manifold (P, Λ) is the cotangent bundle T^*Q, equipped with the Poisson structure associated with its canonic symplectic 2-form Ω. Second, D is the image, by the Legendre transformation, of a submanifold C of TQ, and the vector sub-bundle W of $T_D(T^*Q)$ is contained in the vertical tangent bundle to T^*Q. Therefore the sub-bundle W is isotropic. As a consequence, its symplectic orthogonal, $\text{orth}W$, is coisotropic. We assume that the constrained Hamiltonian system is regular: then $TD \cap W = \{0\}$, and we may consider the direct sum $TD \oplus W$. Let us assume in addition that

$$TD \oplus W = T_D(T^*Q)\,.$$

That condition is fulfilled, for example, when the mechanical system is such that the canonical projection $p : TQ \to Q$, restricted to the constraint submanifold C, is a submersion of C onto an open subset of Q, and that W is the bundle obtained by application of d'Alembert's principle or, more generally, Chetaev's rule. The second part of the regularity condition (that the Hamiltonian vector field X_H, restricted to D, is a section of $TD \oplus W$), is then automatically fulfilled. We have

$$\text{orth}W = \text{orth}W \cap (TD \oplus W) = (\text{orth}W \cap TD) \oplus W\,,$$

since $\text{orth}W \cap W = W$. The kernel of the 2-form induced by the symplectic form Ω on $\text{orth}W$ is W; therefore, $\text{orth}W \cap TD$, which is the complement of W in $\text{orth}W$, is a symplectic vector bundle. This property is used by Bates and Śniatycki [5] for the characterization of the vector field X_D, as the only vector field on D whose values are in the symplectic bundle $\text{orth}W \cap TD$, such that $i(X_D)\Omega + dH$ vanishes on that symplectic bundle. Let us observe, however, that in order to prove that the vector field X_D, whose integral curves are the motions of the system, takes its values in $\text{orth}W \cap TD$, we have to use the fact that dH vanishes on W, in other words, that the mechanical system is conservative. That condition would not be satisfied by more general constraints such as those obtained by servomechanisms; therefore, the characterization of X_D used by Bates and Śniatycki cannot be used in these cases.

3.5 A pseudo-Poisson structure on the Hamiltonian constraint submanifold

Let (P, Λ, H, D, W) be a regular constrained Hamiltonian sytem. We assume in addition that $TD \oplus W = T_D P$; as we have seen in the previous section, this condition is very often fulfilled.

The Poisson tensor Λ of P can be projected on D, and its projection Λ_D is a contravariant skew-symmetric 2-tensor on D. More precisely, let $x \in D$, α and $\beta \in T_x^* D$. There exists a unique pair $(\widetilde{\alpha}, \widetilde{\beta})$ of elements of $T_x^* P$ which vanish on the vector subspace W_x of $T_x P$, and whose restrictions to the vector subspace $T_x D$ are α and β, respectively. We can therefore define $\Lambda_D(x)$ by setting

$$\Lambda_D(x)(\alpha, \beta) = \Lambda(x)(\widetilde{\alpha}, \widetilde{\beta}) .$$

Let H_D be the restriction of H to the submanifold D. Then the vector field X_D, whose integral curves are the motions of the system, is

$$X_D = \Lambda_D^\sharp(dH_D) .$$

In other words, X_D is the unique vector fiel on D such that, for any Pfaff form β on D, we have

$$\Lambda_D(dH_D, \beta) = \langle \beta, X_D \rangle .$$

Using the 2-tensor Λ_D, we can define a composition law $(f, g) \mapsto \{f, g\}_D$ on the space of smooth functions on D, by setting

$$\{f, g\}_D = \Lambda_D(df, dg) .$$

This conposition law is skew-symmetric and it satisfies Leibniz rule. But it is not a true Poisson bracket, since it may not satisfy Jacobi identity. We will say that the corresponding structure on D is a *pseudo-Poisson structure*.

Van der Schaft and Maschke [40] have obtained that pseudo-Poisson structure for a mechanical system with a kinematic constraint linear in the velocities obeying d'Alembert's principle. They have shown that it is a true Poisson structure if and only if the constraint is holonomic. Koon and Marsden [25] use that pseudo-Poisson structure in their theory of Poisson reduction for nonholonomic mechanical systems with symmetry. These authors give a formula which relates the "Jacobiizer" (the left-hand side of the Jacobi identity for that pseudo-Poisson structure) with the non-holonomy of the constraint distribution.

4 Energy, momentum and reduction

4.1 Conservation of energy

In a constrained Hamiltonian system, the Hamiltonian (that means, the total energy of the system) may not remain constant during the motion. This fact is easily understood when the constraint is realized by means of a servomechanism, since that mechanism can give energy to (or withdraw energy from) the mechanical system. For a regular constrained Hamiltonian system, one can easily see that the Hamiltonian H remains constant during every motion of the system if and only if for any kinematic state $v \in C$,

$$\langle X_W(\mathcal{L}(v), v \rangle = 0 ,$$

where $\mathcal{L}$ is the Legendre transformation and X_W the section of W defined in Equation (13). A sufficient condition under which that property holds true is when for each kinematic state $v \in C$, the vector subspace $W(\mathcal{L}(v))$ is contained in the annihilator of v.

4.2 Constrained systems with symmetry

We assume that a symmetry group G acts on a constrained Hamiltonian system (P, Λ, H, D, W) by a Hamiltonian action, leaving invariant all elements of its structure: the Poisson tensor Λ, the Hamiltonian H, the Hamiltonian constraint submanifold D and the bundle W. Let us assume that there exists a momentum map J of that action.

4.2.1 Conservation of momentum

In general, the momentum map J does not remain constant during the motion. A sufficient condition under which the momentum map J remains constant during every motion of the system is $W \subset \ker(TJ)$. When the phase space is a symplectic manifold, $\ker(TJ)$ is the symplectic orthogonal of the bundle tangent to the group orbits.

For the general case, the equations of motion can be arranged in such a way that a separate set of equations describes the evolution of the momentum. In particular, by using a connection, Bloch, Krishnaprasad, Marsden and Murray [9], Koon and Marsden [24, 25] have derived such equations.

4.2.2 Example

We return to Example 3.1.1, where the mechanical system is a convex solid body which rolls without slipping on a horizontal plane F. The group $F \times S^1$ acts on the system by a Hamiltonian action. The corresponding momentum map has as components the horizontal component of the total momentum of the body, and the vertical component of the angular momentum of the body with respect to the origin. It does not remain constant in time.

When the convex body is a dynamically homogeneous sphere, there is another symmetry group, $\mathbf{SO}(3)$, which acts on the system by a Hamiltonian action. The restriction of that action to the subgroup S^1 of rotations around the vertical axis through the center of the sphere satisfies the condition under which the corresponding momentum map remains constant. That momentum map is the angular momentum of the body with respect to that axis.

For a fuller discussion of the eventual conservation of momentum and other examples, the reader is referred to [5, 6, 13, 14]. Equations governing the evolution of momentum are given by Bloch, Krishnaprasad, Marsden and Murray [9], Koon and Marsden [24, 25].

4.2.3 Reduction

We assume that the set $\widehat{P}$ of orbits of the action of G on P has a smooth manifold structure, such that the canonical projection $\pi : P \to \widehat{P}$ is a submersion. One can prove easily that the set $\widehat{D}$ of orbits contained in the submanifold D is a smooth submanifold of $\widehat{P}$. Moreover, since the Hamiltonian H is G-invariant, there exists a unique smooth function $\widehat{H} : \widehat{P} \to \mathbf{R}$ such that $H = \widehat{H} \circ \pi$. The image by π of the vector sub-bundle W of $T_D P$ is a vector sub-bundle $\widehat{W}$ of $T_{\widehat{D}} \widehat{P}$. Moreover, there exists on $\widehat{P}$ a unique Poisson tensor $\widehat{\Lambda}$ such that π is a Poisson map from (P, Λ) onto $(\widehat{P}, \widehat{\Lambda})$. We observe that $(\widehat{P}, \widehat{\Lambda}, \widehat{H}, \widehat{D}, \widehat{W})$ is a constrained Hamiltonian system in the sense of Section 3.4, called the *reduced constrained Hamiltonian system*.

We have shown [30] that the motions of the constrained Hamiltonian system (P, Λ, H, D, W) project onto the motions of the reduced system $(\widehat{P}, \widehat{\Lambda}, \widehat{H}, \widehat{D}, \widehat{W})$, and that if the original system is regular, the reduced system too is regular.

This very straightforward reduction procedure is that used by Hermans [20, 21], also by Bloch, Krishnaprasad, Marsden and Murray [9], and, in a particular case, Koiller [23], although these authors do not all use Poisson structures. Koon and Marsden use Poisson structures and that reduction procedure in [25], and another equivalent reduction procedure, the Lagrangian reduction, in their earlier work [24]. Bates and Śniatycki [5] use a different (but equivalent) reduction procedure, much less straightforward (also used in [6, 13, 14]). On the reduced phase space, they use "partial 2-forms" instead of contravariant 2-tensors, defined as bilinear skewsymmetric forms on a sub-bundle of the tangent bundle to phase space rather than on the whole tangent bundle. We believe that these partial 2-forms correspond to the symplectic 2-forms of the symplectic leaves of the reduced phase space, equipped with its Poisson structure.

The reduction of nonholonomic mechanical systems with symmetry is important for the discussion of stability of stationary motions and the determination of the structure of the set of motions. In that direction, Zenkov [47] has obtained some important results for the motion of a spherical ball which rolls on a surface of revolution; he has shown that the motions are quasi-periodical and fill invariant tori, just like the motions of a completely integrable Hamiltonian system; that result was obtained independently by Hermans [20, 21]. Zenkov has shown that a necessary condition for the stability of stationary motions obtained earlier by Routh is also sufficient.

5 Further developments

The setting for constrained Hamiltonian systems described in Section 3.4 may be not general enough to deal with systems with active kinematic constraints, on which one can act to control the motion of the system. Let us consider again Example 3.1.2 of a convex solid body which rolls without slipping on a horizontal plane which rotates around a vertical axis. But now, let us assume that the angular velocity ω of the rotating plane may vary, and is used as a control parameter on which one may act in order to drive the motion of the system (for example, to stabilize some stationary motion). Instead of a submanifold C of the space of kinematic states TQ, we have now a foliation of a submanifold TQ_1 of TQ, whose different leaves correspond to different values of the angular velocity ω. We plan to develop these ideas in a further work.

6 Acknowledgements

I address my thanks to all the particpants to the workshop on non-holonomic constraints in dynamics for very helpful discussions. R. Cushman helped me to find the paper by Korteweg [26]. Many thanks to the referee for indicating to me several very interesting papers.

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