Geometry of mechanical systems with active and kinematic constraints

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En hommage au Professeur André Lichnerowicz,
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1. Introduction

A mechanical system is said to be *with active constraints* when an operator (either external, or part of the system) can act on some constraints of the system in order to modify its motion. We shall distinguish two classes of mechanical systems with active constraints: systems in which all the constraints are of geometric nature, and systems in which some of the constraints are of kinematic nature. Let us give some examples.

1.1. Systems with geometric constraints only.

1.1.1. A cat in free fall; the cat can act on the shape of his body, in order to modify his orientation in space and reach the ground on his feet.

1.1.2. An artificial satellite, with moving parts such as gyroscopes, antennas or telescopes; an operator can act on these moving parts, and that action may change the orientation of the satellite as a whole.

1.1.3. A child on a swing; by moving his legs and body, he can induce and amplify the oscillations of the swing.

1.2. Systems with geometric and kinematic constraints.

1.2.1. A person riding a bicycle; the rider may act on the pedals and the

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handle-bar, and on the shape and position of his body relative to the bicycle; these constraints, on which he acts, are of geometric nature; the wheels of the bicycle roll on the ground, and that constraint (which is passive, since the rider does not act directly on it) is of kinematic nature.

1.2.2. A car and its driver; the driver can act on the steering wheel, modifying the orientation of the front wheels; such a constraint is of geometric nature; the wheels are rolling on the ground, and that constraint (which is passive) is of kinematic nature.

1.2.3. A hollow sphere rolling on a plane, with somebody inside, walking on the interior wall of the sphere (maybe with magnetic shoes, in order to be able to walk in any position); by modifying his position relative to the sphere, the person inside may act on the overall rolling motion of the sphere; the constraints on which the operator acts directly are of geometric nature (he controls his position relative to the sphere); but the condition that the sphere rolls on the ground wihout slipping is a passive kinematic constraint.

1.2.4. A person skating on ice; he controls directly geometric constraints: the shape of his body, and in particular the relative orientation of the blades of his skates; but at the contact between the skates and the ice, there is a kinematic constraint, which imposes the direction of the motion of the skate relative to the ice.

1.2.5. A convex body rolling on a rotating plate, with an external operator who controls the rotation of that plate. The constraint on which the operator acts (the rotation of the plate) is of geometric nature, but there is also in the system another constraint of kinematic nature (the condition that the convex body is rolling without slipping on the plate), which is passive.

2. Mechanical systems with active geometric constraints

2.1. The configuration and the constraint state spaces.

We consider in this section a mechanical system with an active geometric constraint. As in [6,7], we will take a smooth manifold M as configuration space of the system, another smooth manifold S as space of possible states of the active constraint. We assume that the map which associates, with each configuration of the system, the corresponding state of the active constraint is a surjective submersion $\pi : M \to S$. The space of kinematical states of the system is the tangent bundle TM to the configuration space M. The dynamical properties of the system are mathematically described by a smooth function $L : TM \to \mathbf{R}$, the *Lagrangian* of the system. We shall assume that the only forces which act on the system not already accounted for by the Lagrangian, are the forces exerted by the active constraint.

2.2. Lagrange's equations in local coordinates.

Let $\dim M = m$, $\dim S = m - p \geq 0$. We choose a chart of M and a chart of S adapted to the submersion $\pi : M \to S$; it means that in the local coordinates (x^i, x^a) on M, s^a on S, associated with the chosen charts, the submersion π is expressed as

$$\pi : (x^i, x^a) \mapsto s^a = x^a\,, \quad 1 \leq i \leq p\,, \quad p+1 \leq a \leq m\,.$$

Let (x^i, x^a, v^i, v^a) be the local coordinates in the associated chart of TM. The equations of motion may be written as

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial L}{\partial v^i}\right) - \frac{\partial L}{\partial x^i} &= 0\,, \qquad 1 \leq i \leq p\,,\\
\frac{d}{dt}\left(\frac{\partial L}{\partial v^a}\right) - \frac{\partial L}{\partial x^a} &= \lambda_a\,, \qquad p+1 \leq a \leq m\,,
\end{aligned}$$

where the Lagrange multipliers λ_a correspond to the force exerted by the active constraint.

2.3. Hamilton's equations in local coordinates.

Let T^*M (the cotangent bundle of the configuration manifold M) be the phase space of the system, and $\mathcal{L} : TM \to T^*M$ be the Legendre transformation associated with the Lagrangian L (see for exxample W. Tulczyjew [10]). Let (x^i, x^a, p_i, p_a) ($1 \leq i \leq p$, $p+1 \leq a \leq m$) be the local coordinates in the chart of T^*M associated with the chosen chart of M. In these local coordinates, the map $\mathcal{L}$ may be expressed as

$$\mathcal{L} : (x^i, x^a, v^i, v^a) \mapsto (x^i, x^a, p_i, p_a)\,, \quad \text{with } p_i = \frac{\partial L}{\partial v^i}\,,\ p_a = \frac{\partial L}{\partial v^a}\,.$$

We will assume that $\mathcal{L}$ is a diffeomorphism. Let $Z = \sum_{k=1}^{m} v^k \dfrac{\partial}{\partial v^k}$ be the Liouville vector field on TM, and $H = \big(i(Z)dL - L\big) \circ \mathcal{L}^{-1}$ be the Hamiltonian associated with the Lagrangian L. The equations of motion, in Hamilton's formalism, may be written as

$$\frac{dx^i}{dt} = \frac{\partial H}{\partial p_i}\,, \qquad 1 \leq i \leq p\,, \tag{1}$$

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial x^i}\,, \qquad 1 \leq i \leq p\,, \tag{2}$$

$$\frac{\partial H}{\partial p_a} = \frac{ds^a}{dt}\,, \qquad p+1 \leq a \leq m\,, \tag{3}$$

$$\lambda_a = \frac{dp_a}{dt} + \frac{\partial H}{\partial x^a}\,, \qquad p+1 \leq a \leq m\,. \tag{4}$$

Let us observe that in Equation (3), the right hand side $\frac{ds^a}{dt}$ should be considered as a known quantity: it is the velocity with which the operator is changing the state of the active constraint. Therefore, Equation (3) should be used in order to determine the p_a $(p+1 \le a \le m)$ in terms of the x^i, x^a, p_i and $\frac{ds^a}{dt}$ $(1 \le i \le p)$. The expressions of the p_a obtained by that mean can then be fed into Equations (1) and (2). If we assume that the action of the operator is known as a function of time, the x^a $(p+1 \le a \le m)$ are known functions of time, and we obtain, by that mean, a nonautonomous differential system for the x^i and p_i $(1 \le i \le p)$. After solving that system, Equation (4) should be used to calculate the constraint forces λ_a $(p+1 \le a \le m)$ as functions of time.

2.4. An intrinsic formulation.

The process described above in local coordinates can be formulated in an intrinsic manner, by using the concept of Poisson manifold, introduced by A. Lichenrowicz [5]. For the properties of Poisson manifolds used here, the reader is referred to A. Weinstein [14]. Let $VM = \ker(T\pi) \subset TM$ be the vertical bundle, that means the set of vectors tangent to the configuration manifold M, whose projection on S vanish. The dual bundle of VM is the quotient bundle $V^*M = T^*M/(VM)^0$, where $(VM)^0$ is the annihilator of VM, that means the set of covectors on M which vanish on the vertical bundle VM. We denote by $\zeta : T^*M \to V^*M$ and $q : V^*M \to M$ the canonical projections, and we set $\widetilde{\pi} = \pi \circ q : V^*M \to S$. Then V^*M is a Poisson manifold. We will assume, for simplicity, that for each $s \in S$, $M_s = \pi^{-1}(s)$ is connected; it is the configuration space of the system when the state of the active constraint is $s \in S$. The symplectic leaves of the Poisson manifold V^*M are the fibres $\widetilde{\pi}^{-1}(s)$, for all $s \in S$; each fibre $\widetilde{\pi}^{-1}(S)$ may be identified, by a canonical symplectic diffeomorphism, with the cotangent space T^*M_s. The manifold S appears as the space of symplectic leaves of the Poisson manifold V^*M.

Let $s(t) \in S$ be the state of the active constraint at time t. We assume, for simplicity, that a vector field Y is given on S and that the operator acts on the active constraint in such a way that $t \mapsto s(t)$ is an integral curve of Y. By solving Equation (3), in order to determine the p_a $(p+1 \le a \le m)$ in terms of the x^a, x^i, p_i $(1 \le i \le p)$, we define an immersion χ of V^*M (with x^i, x^a, p_i as local coordinates) into T^*M (with x^i, x^a, p_i, p_a as local coordinates).

The time evolution of the state of the system is then described by an integral curve of a vector field D_Y on V^*M, obtained by the following process:

— we take the Hamiltonian vector field X_H on T^*M, associated with the Hamiltonian H;

— we take the restriction of X_H to the image $\chi(V^*M)$ of the immersion $\chi : V^*M \to T^*M$ defined above;

— we project that restriction onto V^*M, by the canonical projection $\zeta : T^*M \to V^*M$.

That projection is the vector field D_Y on V^*M, whose integral curves describe the time evolution of the system. We will say that D_Y is the *dynamical vector field* associated with Y. It is easy to verify that D_Y projects onto S and has the vector field Y as its projection.

For any vector field Y on S, the corresponding dynamical vector field D_Y on V^*M is a Poisson infinitesimal automorphism of V^*M. For $Y = 0$, the corresponding dynamical vector field D_0 is a Hamiltonian vector field on the Poisson manifold V^*M; therefore it is tangent to the symplectic leaves.

2.5. A canonical decomposition of the evolution vector field.

We assume now that our system is a classical mechanical sytem; it means that its Lagrangian L can be written as

$$L(x,v) = \frac{1}{2} g_x(v,v) - P(x)\,, \qquad x \in M\,, \quad v \in T_xM\,,$$

where g is a Riemannian metric and P a smooth function on M. We assume, as in the previous section, that a vector field Y is given on S, and that the operator acts on the active constraint in such a way that the time evolution of the state of the constraint is an integral curve of Y.

By using the Riemannian metric g, we define a symmetric bilinear form on the vector bundle $\pi^*(TS)$, inverse image of the tangent bundle to S by the submersion $\pi : M \to S$. That bilinear form is given by

$$\Theta_x(\pi^* u_1, \pi^* u_2) = g_x(v_1, v_2)\,, \qquad x \in M\,, \quad u_1 \text{ and } u_2 \in T_{\pi(x)}S\,,$$

where v_1 and v_2 are two vectors tangent to M at x, which are orthogonal (for the Riemannian metric g) to the vertical subspace $V_xM = \ker(T_x\pi)$, and such that $T_x\pi(v_1) = u_1$, $T_x\pi(v_2) = u_2$.

We define now a smooth function K_Y on M by setting

$$K_Y(x) = \frac{1}{2}\Theta_x\bigl(\pi^*(Y_{\pi(x)}), \pi^*(Y_{\pi(x)})\bigr)\,.$$

The function K_Y may be considered, in some sense, as the kinetic energy of the active constraint. We will still denote by $K_Y : V^*M \to \mathbf{R}$ the composition of K_Y with the canonical projection $q : V^*M \to M$.

It may be proven [7] that the vector field D_Y canonically splits into a sum of three terms,

$$D_Y = D_0 - X_{K_Y} + H(Y)\,.$$

The first term D_0 is the vector field which describes the time evolution of the system when the state of the active constraint remains constant in time (in other terms, it is the vector field D_Y when $Y = 0$). The second term X_{K_Y} is the Hamiltonian vector field on the Poisson manifold V^*M, whose Hamiltonian function is K_Y. The third term $H(Y)$ is the horizontal lift of the vector field Y, for an Ehresmann connection on the bundle $\widetilde{\pi} : V^*M \to S$. That Ehresmann connection, called the *dynamical connection*, is completely determined by the Riemannian metric on M which describes the kinetic energy, and by the submersion $\pi : M \to S$.

The dynamical connection is the only Ehresmann connection on the bundle $\widetilde{\pi} : V^*M \to S$ which satisfies the following two properties:

— for any $z \in V^*M$ and $u \in T_{\widetilde{\pi}(z)}S$, the horizontal lift $H(u)$ of u at z, for the dynamical connection, is such that its projection $T_z q\big(H(u)\big)$ on M, is the unique vector tangent to M at $x = q(z)$, whose projection on S is u, and which is orthogonal (for the Riemannian metric g) to the vertical subspace $V_xM = \ker T_x\pi$;

— for any vector field Y on S, the horizontal lift $H(Y)$ of Y, relative to the dynamical connection, is an infinitesimal automorphism of the Poisson manifold V^*M, tangent to the zero section of the bundle $q : V^*M \to M$.

3. Systems with geometric and kinematic constraints; the easiest case

Let us now look at systems with geometric and kinematic constraints, in which the active constraint is of geometric nature. Systems described in Examples 1.2.1 to 1.2.5 are of that kind.

3.1. Mathematical description of a kinematic constraint.

Following Faddeev and Vershik [11,12,13], as in [8], we will describe a kinematic constraint by a submanifold C of the tangent bundle TM; that submanifold is the space of admissible kinematical states, that means, kinematical states for which the kinematical constraint is satisfied. We will say that C is the *Lagrangian constraint submanifold*. Let $p : TM \to M$ be the canonical projection. We will make the following assumptions:

— the projection $M_1 = p(C)$ of the constraint submanifold C is a submanifold of M;

— the canonical projection restricted to the constraint submanifold, $p \big|_C$, is a submersion of C onto M_1;

— the constraint submanifold C is contained in TM_1.

Let $D = \mathcal{L}(C) \subset T^*M$ be the image of the Lagrangian constraint submanifold C by the Legendre transformation. We will say that D is the *Hamiltonian constraint submanifold*.

The knowledge of the Lagrangian constraint submanifold C, or of the Hamiltonian constraint submanifold D, is not sufficient to determine all the mechanical properties of the constraint: we must, in addition, indicate what are the possible values of the constraint force.

3.1.1. Passive linear kinematic constraints.

In most cases encountered in mechanics, the constraint submanifold C associated with a kinematic constraint is an affine subbundle of TM_1; this happens, for example, for mechanical systems in which two rigid bodies are rolling without slipping on each other (see [3,4] for a thorough investigation of rolling rigid bodies). With the assumptions made above, for each point $x \in M_1$, $C_x = C \cap p^{-1}(x)$ is an affine subspace of T_xM_1, which is itself a vector subspace of T_xM. Therefore, C_x is an affine subspace of T_xM. Let $\overrightarrow{C_x}$ be the associated vector subspace. When the constraint under consideration is passive (it means that no operator acts on it) and perfect, the possible values of the constraint force are determined by the following condition: when the configuration of the system is a point $x \in M_1$, the constraint force must belong to the annihilator $(\overrightarrow{C_x})^0$ of $\overrightarrow{C_x}$, that means, to the vector subspace of T^*_xM made by covectors which vanish on the vector subspace $\overrightarrow{C_x}$ of T_xM. That condition is the well known *d'Alembert's principle* of virtual work: when a constraint is perfect, its virtual work, for any infinitesimal virtual displacement compatible with the constraint frozen in its state at the time under consideration, must vanish.

3.1.2. Passive nonlinear kinematic constraints.

In [8], we proposed the following way of expressing d'Alembert's principle for a passive, perfect kinematic constraint, when the Lagrangian constraint submanifold C may not be an affine subbundle of TM_1. Such a constraint will be said to be *nonlinear* (in the velocities). For each point $x \in M_1$, C_x is a submanifold of T_xM_1, which is itself a vector subspace of T_xM; therefore, C_x is a submanifold of T_xM. Let us assume that the kinematic state of the system, at the time under consideration, is a point $v \in C_x$. The tangent space T_vC_x is a vector subspace of $T_v(T_xM)$. But since T_xM is a vector space, the tangent space to T_xM an any point, and in particular at point v, can be identified with T_xM itself. Therefore, T_vC_x can be considered as a vector subspace of T_xM. We now express d'Alembert's principle by the following condition: when the kinematical state of the system is v, the constraint force must belong to the annihilator $(T_vC_x)^0$ of T_vC_x, that means, to the vector subspace of T^*_xM made by covectors which vanish on T_vC_x, considered as a vector subspace of T_xM.

When the Lagrangian constraint submanifold C is an affine subbundle of TM_1, this condition is equivalent to that given by the the usual form of d'Alembert's principle, indicated in Section 3.1.1. For that reason we shall use it in the following. However, it is not quite clear, at least for the author, whether the above condition is, in all cases, a physically valid mathematical description of passive, perfect, nonlinear kinematic constraints. P. Dazord [2] has introduced still another method for the treatment of (active or passive) kinematic constraints, which will not be used here. Many other authors have developed various approaches for the mathematical modelling of kinematic constraints, in particular S. Benenti [1], J. Hermans [3,4], E. Massa and E. Pagani [9].

For systems with passive constraints only, we have shown [8] that the above condition leads to a well defined differential equation, which describes the time evolution of the system. That differential equation is obtained by the following process. For each $x \in M_1$ and $v \in C_x$, let $p = \mathcal{L}(v) \in D_x$ be the image of v by the Legendre transformation. We recall that $q : T^*M \to M$ denotes the canonical projection. The vertical subspace $\ker T_p q = T_p(T^*_x M)$ of the tangent space to T^*M at p can be canonically identified with $T^*_x M$, since it is the space tangent to a vector space $T^*_x M$ at one of its points. Therefore, the the annihilator $(T_v C_x)^0$ of $T_v C_x$ can be identified with a vector subspace W_p of $T_p(T^*M)$. We have therefore associated with each point p of the Hamiltonian constraint submanifold D, a vector subspace W_p of $T_p(T^*M)$, contained in the vertical subpace $\ker T_p q$. It can be shown that the vector subbundle W of $T_D(T^*M)$ so defined is smooth, and such that $W \cap TD = \{0\}$. Moreover, the Hamiltonian vector field X_H on T^*M, restricted to the Hamiltonian constraint submanifold D, is contained in the direct sum $W \oplus TD$. We can therefore, in a unique way, split $X_H \big|_D$ into a sum

$$X_H \big|_D = X_D + X_W \,,$$

where X_D is a vector field on D, and X_W a smooth section of the vector bundle W.

The time evolution of the system is governed by the differential equation associated with the vector field X_D. The force exerted by the kinematic constraint is given by $-X_W$.

3.2. Systems with a passive kinematic constraint and an active geometric constraint.

When, in addition to the passive kinematic constraint, we have an active geometric constraint, the Hamiltonian description of the system is the following. The Hamiltonian is a smooth function $H : T^*M \to \mathbf{R}$ on the phase space T^*M of the system. We have a surjective submersion $\pi : M \to S$, with connected fibres, of the configuration manifold M onto the manifold S of states of the active constraint. The Hamiltonian constraint submanifold is a submanifold D of T^*M, whose projection on M is a submanifold M_1 of M. Along D, the vector subbundle W of $T_D(T^*M)$, defined as indicated above, is such that $W \cap TD = \{0\}$ and that $X_H \big|_D$ is a section of $TD \oplus W$. For simplicity, let us assume, as above, that a vector field Y is given on S, and that the operator acts on the active constraint in such a way that the time evolution of the state of that constraint follows an integral curve of Y. This enables us to define, as in 2.4, an immersion $\chi : V^*M \to T^*M$. Under some natural assumptions, it can be shown that $\chi(V^*M)$ intersects D transversally. Let us now take the restriction of the Hamiltonian vector field X_H to the submanifold $\chi(V^*M) \cap D$. As above, we can split it, in a unique way, into a sum of a vector field X_D, tangent to the submanifold $\chi(V^*M) \cap D$, and a section X_W of the vector bundle $W \big|_{\chi(V^*M)\cap D}$. Finally, the vector field whose integral curves describe the time evolution of the system is the projection of X_D on V^*M.

3.3. Example: a simple model of skating on ice.

We consider a system made of two rigid bodies, with a common vertical axis of symmetry, which may rotate around that axis relative to each other. The lower body rests on a horizontal plane and may slide on that plane in a direction fixed with respect to the solid body.

Such a system is a very rough representation of a skater on ice: the lower body represents his legs equipped with skates (assumed to be always parallel), and the upper body his chest, shoulders, arms and head. The skater can rotate around a vertical axis his head, chest shoulders and arms, relative to his legs and skates.

The state of the system can be parametrized by the two linear coordinates x and y of the point at which the vertical symmetry axis of the system intersects the horizontal plane, and the two angular coordinates θ_1 and θ_2 which measure the orientation of the two bodies, relative to the Ox axis. Therefore, the configuration manifold is $M = \mathbf{R}^2 \times \mathbf{T}^2$. The manifold S is $\mathbf{S}^1$, and the projection $\pi : M \to S$ is the map $(x, y, \theta_1, \theta_2) \mapsto s = \theta_2 - \theta_1$. We will take x, y, θ_1 and $s = \theta_2 - \theta_1$ as coordinates on M, so they are adapted to the submersion π. On the fibres of the tangent and cotangent bundles, TM and T^*M, the corresponding coordinates will be denoted by v_x, v_y, v_{θ_1}, v_s and p_x, p_y, p_{θ_1}, p_s, respectively.

The Lagrangian of the system is

$$L = \frac{1}{2}m(v_x^2 + v_y^2) + \frac{1}{2}I_1 v_{\theta_1}^2 + \frac{1}{2}I_2(v_{\theta_1} + v_s)^2 \,,$$

where m is the total mass of the system, I_1 and I_2 the moments of inertia of the two bodies with respect to their vertical symmetry axis.

The Legendre transformation $\mathcal{L} : TM \to T^*M$, is the map

$$\mathcal{L} : (x, y, \theta_1, s, v_x, v_y, v_{\theta_1}, v_s) \mapsto (x, y, \theta_1, s, p_x, p_y, p_{\theta_1}, p_s) \,,$$

where

$$p_x = m v_x \,, \quad p_y = m v_y \,, \quad p_{\theta_1} = (I_1 + I_2) v_{\theta_1} + I_2 v_s \,, \quad p_s = I_2(v_{\theta_1} + v_s) \,.$$

The Hamiltonian of the system is

$$H = \frac{1}{2m}(p_x^2 + p_y^2) + \frac{1}{2I_1}(p_{\theta_1} - p_s)^2 + \frac{1}{2I_2} p_s^2 \,.$$

The Lagrangian constraint submanifold $C \subset TM$ is defined by

$$v_x \sin\theta_1 - v_y \cos\theta_1 = 0 \,,$$

and the Hamiltonian constraint submanifold $D \subset T^*M$ by

$$p_x \sin\theta_1 - p_y \cos\theta_1 = 0 \,.$$

The projection bundle W along D is of rank 1; it is generated by the vector field, denoted by the same letter W,

$$W = \sin\theta_1 \frac{\partial}{\partial p_x} - \cos\theta_1 \frac{\partial}{\partial p_y} \,.$$

Let

$$\frac{ds}{dt} = Y(s)$$

be the differential equation which governs the time evolution of the state of the active constraint. We assume that the vector field Y on S is known. The coordinates on V^*M are $(x, y, \theta_1, s, p_x, p_y, p_{\theta_1})$, and the immersion $\chi : V^*M \to T^*M$ is obtained by using the following component of Hamilton's equation,

$$\frac{ds}{dt} = Y(s) = \frac{\partial H}{\partial p_s} = \frac{I_1 + I_2}{I_1 I_1} \, p_s - \frac{1}{I_1} \, p_{\theta_1} \, ,$$

in order to calculate p_s in terms of the coordinates on V^*M. We obtain

$$p_s = \frac{I_2}{I_1 + I_2} \, p_{\theta_1} + \frac{I_1 + I_2}{I_1 I_2} \, Y(s) \, .$$

The Hamiltonian vector field X_H is

$$X_H = \frac{1}{m} \, p_x \, \frac{\partial}{\partial x} + \frac{1}{m} \, p_y \, \frac{\partial}{\partial y} + \frac{1}{I_1} \, (p_{\theta_1} - p_s) \, \frac{\partial}{\partial \theta_1} + \left(\frac{I_1 + I_2}{I_1 I_2} \, p_s - \frac{1}{I_1} \, p_{\theta_1} \right) \frac{\partial}{\partial s} \, .$$

We must consider the vector field $X_H + \lambda W$, and choose λ in such a way that it is tangent to $\chi(V^*M) \cap D$. We obtain

$$\lambda = (p_x \cos\theta_1 + p_y \sin\theta_1) \left(\frac{I_2}{I_1 + I_2} \, Y(s) - \frac{1}{I_1 + I_2} \, p_{\theta_1} \right) \, .$$

The evolution equations of the system are obtained by projecting onto V^*M the vector field $X_H + \lambda W$, restricted to $\chi(V^*M) \cap D$, with λ given by the above equation:

$$\frac{dx}{dt} = \frac{1}{m} \, p_x \, , \qquad \frac{dp_x}{dt} = \lambda \sin\theta_1 \, ,$$

$$\frac{dy}{dt} = \frac{1}{m} \, p_y \, , \qquad \frac{dp_y}{dt} = -\lambda \cos\theta_1 \, ,$$

$$\frac{d\theta_1}{dt} = \frac{1}{I_1 + I_2} \, p_{\theta_1} - \frac{I_2}{I_1 + I_2} \, Y(s) \, , \qquad \frac{dp_{\theta_1}}{dt} = 0 \, .$$

$$\frac{ds}{dt} = Y(s) \, ,$$

We observe that

$$p_{\theta_1} = (I_1 + I_2) \frac{d\theta_1}{dt} + I_2 Y(s)$$

is a first integral of the system. Moreover, let us set $p = \sqrt{p_x^2 + p_y^2}$. Then $p_x = p \cos\theta_1$, $p_y = p \sin\theta_1$, and we observe that p is another first integral of the system.

By using these two first integrals, the system can be explicitely solved by quadratures.

4. Systems with active kinematic constraints

The mathematical model described above is inadequate for systems in which the operator acts directly on a kinematic constraint, although the mechanical systems of that kind encountered in practice are limits, obtained by neglecting some masses or moments of inertia, of systems with a passive kinematic constraint and an active geometric constraint. We indicate below a possible approach for modelling such systems, and we describe a very simple example. We do not claim that our approach covers all the cases of interest. For a completely different approach of that problem, the reader is referred to P. Dazord [2].

4.1. The mathematical model.

We use a smooth manifold M as configuration state of the system. Since the active constraint is now of kinematic nature, it cannot be modelled by a surjective submersion of M onto another manifold. We use instead a submanifold C of the tangent bundle TM, foliated into leaves C_s, the leaves being parametrized by points s of a manifold S. The submanifold C of TM is the set of all possible kinematical states of the system, for all possible states of the constraint. The manifold S is the space of all possible states of the active constraint, and for a given point $s \in S$, the leaf C_s of the foliated manifold C is the set of possible kinematic states of the system when the state of the active constraint is s.

In all examples known to the author, the submanifold C of TM is an open dense subset of an affine subbundle of TM, and the leaves C_s, $s \in S$, of the foliation of C, are open subsets of affine subbundles of TM, of rank smaller than the rank of C.

The dynamical properties of the system are given by a smooth function $L : TM \to \mathbf{R}$, the Lagrangian of the system. As above we will assume that the corresponding Legendre transformation $\mathcal{L} : TM \to T^*M$ is a diffeomorphism, and we will denote by $H : T^*M \to \mathbf{R}$ the Hamiltonian. The image $D = \mathcal{L}(C)$ of the submanifold C by the Legendre transformation is a submanifold of T^*M, foliated into leaves $D_s = \mathcal{L}(C_s)$, $s \in S$.

We will assume for simplicity that there exists a surjective submersion $\pi : D \to S$, whose fibres $\pi^{-1}(s)$, $s \in S$, are the leaves D_s. As above, we will assume that a vector field Y is given on S, and that the operator action is such that the time evolution of the active constraint is an integral curve of Y.

For each $s \in S$, the method described is section 3.1.2 may be used, with D_s instead of D. We obtain, by that mean, a vector subbuundle W_s of $T_{D_s}(T^*M)$. The union of the vector bundles W_s, $s \in S$, is a vector subbundle W of $T_D(T^*M)$.

Now we consider the Hamiltonian vector field X_H on T^*M. In most examples, its restriction to D splits, in a unique way, into a sum of a vector field X_D tangent to D, whose projection onto S is Y, and a section X_W of W. The time evolution of the system is given by integral curves of X_D, and for each time t, the corresponding value of $-X_W$ is the force exerted by the active constraint.

4.2. A simple example.

Let us consider a simple example, which is another very rough model of skating on ice, different from that described in Section 3.3. This model aims to describe the motion of a skater who puts one of his skate just before the other, and who acts on the angle made by the blades of his two skates. As a result, the skater controls, as a function of time, the radius of curvature of the line that he is drawing on the ice.

The configuration manifold is $M = \mathbf{R}^2 \times \mathbf{S}^1$; the linear coordinates x and y on $\mathbf{R}^2$ are those of the position of the skater, and $\theta \in \mathbf{S}^1$ is is the angle made by the direction in which the skater is sliding with the Ox axis. The corresponding coordinates on the fibres of TM are denoted by v_x, v_y and v_θ, and those one the fibres of T^*M by p_x, p_y, p_θ. The submanifold C of TM is defined by

$$v_x \sin\theta - v_y \cos\theta = 0\,, \qquad v_x^2 + v_y^2 \neq 0\,.$$

For a given value s of the curvature, the leaf C_s of C is defined by the additional equation

$$v_\theta - s(v_x \cos\theta + v_y \sin\theta) = 0\,.$$

The Hamiltonian of the system is

$$H = \frac{1}{2m}\,(p_x^2 + p_y^2) + \frac{1}{2I}\,p_\theta^2\,,$$

where m is the mass and I the moment of inertia of the system with respect with the vertical axis through its center of mass. The Legendre transformation $\mathcal{L}$ is therefore defined by

$$p_x = mv_x\,, \quad p_y = mv_y\,, \quad p_\theta = Iv_\theta\,.$$

The submanifold D of T^*M is defined by

$$p_x \sin\theta - p_y \cos\theta = 0\,, \qquad p_x^2 + p_y^2 \neq 0\,.$$

For each value s of the curvature, the leaf D_s of D is defined by the additional equation

$$mp_\theta - Is(p_x \cos\theta + p_y \sin\theta) = 0\,.$$

The vector subbundle W of $T_D(T^*M)$ is generated by the two vector fields along D:

$$\sin\theta\,\frac{\partial}{\partial p_x} - \cos\theta\,\frac{\partial}{\partial p_y}\,, \qquad \frac{\partial}{\partial p_\theta} - s\left(\cos\theta\,\frac{\partial}{\partial p_x} + \sin\theta\,\frac{\partial}{\partial p_y}\right)\,.$$

The equations of motion on T^*M are

$$\begin{aligned}
\frac{dx}{dt} &= \frac{1}{m}p_x\,, & \frac{dp_x}{dt} &= \lambda\sin\theta - \mu s\cos\theta\,,\\
\frac{dy}{dt} &= \frac{1}{m}p_y\,, & \frac{dp_y}{dt} &= -\lambda\cos\theta - \mu s\sin\theta\,,\\
\frac{d\theta}{dt} &= \frac{1}{I}p_\theta\,, & \frac{dp_\theta}{dt} &= \mu,\,,
\end{aligned}$$

where the coefficients λ and μ must be chosen in such a way that the equation which defines D remains satisfied for all times, and that $\dfrac{ds}{dt}$ is a given function of time. By setting $p = \sqrt{p_x^2 + p_y^2}$, we obtain $p_x = p\cos\theta$, $p_y = p\sin\theta$, and we obtain

$$\mu = \frac{Ip}{m + Is^2}\,\frac{ds}{dt}\,, \qquad \lambda = -\frac{p}{I}\,p_\theta\,.$$

As indicated in the previous section, we obtain a well defined differential equation on D. We observe that the energy H is a first integral of the system.

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